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§ 2. *Estimate of the difference of the integrals of one term of a superposition along nearby level curves*

Let G be a region of the plane of the variables x and y, and $q_1(x, y)$ and $q_2(x, y)$ continuously differentiable functions satisfying in this region the following conditions: a) the partial derivatives with respect to x and with respect to y have modulus of continuity $\omega(\delta)$; b) the inequalities

$$0 < \gamma \leqslant |\ \mathrm{grad}\ [q_i(x, y)]\ | \leqslant \frac{1}{\gamma} < \infty \qquad (i = 1, 2)$$

are satisfied everywhere in G, where γ is a constant; c) for any point $(x, y) \in G$ the absolute value of the acute angle formed by the level curves of the functions $q_1(x, y)$ and $q_2(x, y)$ which pass through this point is greater than some positive constant γ.

LEMMA 4.2.1. *Let e'_{q2} and e''_{q2} be two level curves of the function q_2 and e'_{q1} and e''_{q1} level curves of the function q_1; $[a', a''] \subset G$ the segment of the curve e'_{q1} with end-points $a' \in e'_{q2}$ and $a'' \in e''_{q2}$; $[b', b'']$ the segment of the curve e''_{q1} with end-points $b' \in e'_{q2}$ and $b'' \in e''_{q2}$. Then*

$$h_1([b', b'']) \leqslant h_1([a', a'']) \times (1 + c_1(\gamma)\,\omega(\delta)),$$

where $\delta = d_1([a', a''] \cup [b', b''])$ and $c_1(\gamma)$ depends only on γ.

Proof. Since $q_2(a'') - q_2(a') = q_2(b'') - q_2(b')$, we have

$$\int\limits_{s \in [a', a'']} \frac{\partial q_2}{\partial s}\, ds = \int\limits_{s \in [b', b'']} \frac{\partial q_2}{\partial s}\, ds .$$

Consequently, $\dfrac{\partial q_2(a^*)}{\partial s} h_1([a', a'']) = \dfrac{\partial q_2(b^*)}{\partial s} h_1([b', b''])$, where $\dfrac{\partial q_2(a^*)}{\partial s}$ and $\dfrac{\partial q_2(b^*)}{\partial s}$ are the derivatives at the points $a^* \in [a', a'']$ and $b^* \in [b', b'']$ along the curves $[a', a'']$ and $[b', b'']$, respectively. We show that $\dfrac{\partial q_2(a^*)}{\partial s} = \dfrac{\partial q_2(b^*)}{\partial s} + O(\gamma)\,\omega(\delta)$. We denote by q_2^* the derivative of q_2 at the point b^* in the direction of $\tau(e'_{q1}, a^*)$ and put $\alpha = \gamma\{\tau[e''_{q1}, b^*], \tau[e'_{q1}, a^*]\}$. From conditions a) and b) it follows that $\dfrac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1)\,\omega(\delta)$ and α

$= O(\gamma)\,\omega(\delta)$. We denote by β_1 and β_2 the values of the angles formed by the vectors $\tau\,[e''_{q_1}, b^*]$ and $\tau\,[e'_{q_1}, a^*]$ with the vector grad $[q_2(b^*)]$. We have

$$\left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| = |\operatorname{grad}[q_2(b^*)]|\;|\cos\beta_2 - \cos\beta_1| = O(\gamma)\,\alpha$$
$$= O(\gamma)\,\omega(\delta)\,.$$

Thus,

$$\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1)\,\omega(\delta) = \frac{\partial q_2(b^*)}{\partial s}$$
$$+ O(1)\left\{\left| q_2^* - \frac{\partial q_2(b^*)}{\partial s}\right| + \omega(\delta)\right\} = \frac{\partial q_2(b^*)}{\partial s} + O(\gamma)\,\omega(\delta)\,.$$

Consequently,

$$h_1([b', b'']) = h_1([a', a''])\,\frac{\partial q_2(a^*)}{\partial s}\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1}$$
$$= h_1([a', a''])\left(1 + O(\gamma)\,\omega(\delta)\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1}\right)$$
$$= h_1([a', a''])\,(1 + O(\gamma)\,\omega(\gamma))\,,$$

since by virtue of b) $\dfrac{\partial q_2(b^*)}{\partial s} > |\operatorname{grad}[q_2(b^*)]|\sin\gamma$. This, proves the lemma.

Lemma 4.2.2. *Let $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ be continuously differentiable functions. In any region D we can find a subregion $G \subset D$, determine a constant $\gamma > 0$, and renumber the functions $\{q_m(x, y)\}$ with two indices so that the functions*

$$q_i^k(x, y) = q_m(x, y) \quad \left(i = 0, 1, 2, \ldots, n\,;\; k = 1, 2, \ldots, m_i\,;\; \sum_{i=0}^{n} m_i = N\right)$$

obtained after the renumbering satisfy the following conditions:

(1) *when* $i = 0$, $q_i^k = \text{const}$ *in* G, *and when* $i > 0$, $\gamma \leqslant |\operatorname{grad}[q_i^k(x, y)]| \leqslant \dfrac{1}{\gamma}$ *for every point* $(x, y) \in G$;

(2) *the functions* $q_i^k(x, y)$ $(i > 0$ *fixed,* $k = 1, 2, \ldots, m_i)$ *have in the region* G *identical sets of level curves, more precisely, in the region* G, $q_i^k(x, y) \equiv \varphi_i^{k,l}(q_i^l(x, y))$, *where* $\varphi_i^{k,l}(t)$ *is a strictly monotonic continuously differentiable function of* t;

(3) *when* $i \neq j\,(i, j \neq 0)$, *then for any* k *and* l *the absolute value of the acute angle formed by the level curves of the functions* $q_i^k(x, y)$ *and* $q_j^l(x, y)$ *which pass through an arbitrary point* $(x, y) \in G$ *is greater than* γ.

Proof. By the continuity of the partial derivatives of the functions $\{ q_m(x, y) \}$ there exists a subregion $G^* \subset D$ inside which for any function $q_m(x, y)$ either $\operatorname{grad} q_m(x, y) \equiv 0$ or $|\operatorname{grad} q_m(x, y)|$ is greater than some positive constant. From the continuity of the partial derivatives of the functions $\{ q_m(x, y) \}$ it follows also that there exists a subregion $G^{**} \subset G^*$ inside which for any pair of functions $q_r(x, y)$ and $q_s(x, y)$ one of two conditions holds: either $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$ in G^{**}, or for every point of G^{**} the level curves of $q_r(x, y)$ and $q_s(x, y)$ that pass through this point intersect at a non-zero angle $\left(D\left(\frac{q_r, q_s}{x, y}\right) \neq 0 \text{ in } G^{**}\right)$. From the implicit function theorem it follows that there exists a subregion $G \subset G^{**}$ in which condition (2) is satisfied for every pair of functions $q_r(x, y)$ and $q_s(x, y)$ with gradients different from zero and with determinant $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$.

We now renumber the functions $\{ q_m(x, y) \}$ with two indices in such a way that only functions constant in G have lower index zero, and the same lower index is assigned to those functions whose level curves coincide identically in G. This proves the lemma.

We consider in the region G a superposition of the form $\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_r(x, y)$ $f_i^k(q_i^k(x, y))$, where $\{f_i^k(t)\}$ are continuous functions of one variable, $\{ p_i^k(x, y) \}$ are continuous functions satisfying in G the condition $|p_i^k(x, y)| \leqslant \frac{1}{\gamma}$ and $\{ q_i^k(x, y) \}$ are continuously differentiable functions satisfying in G conditions (1), (2), (3) of Lemma 4.2.2. Let $\omega(\delta)$ be the common modulus of continuity in G of the functions $\left\{p_i^k(x, y);\ \frac{\partial q_i^k(x, y)}{\partial x};\ \frac{\partial q_i^k(x, y)}{\partial y}\right\}$. Let $[a', a'']$ and $[b', b'']$ be segments of the level curves of the functions $\{ q_i^k(x, y) \}$ ($i > 0$ fixed) lying in G. Let

$$\alpha = h_1([a', a'']);\ \delta = \rho([a', a''],\ [b', b'']);$$

$$\varepsilon = \sup \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right|;$$

$$m = \max_{i,k} \sup \left| f_i^k (q_i^k (x, y)) \right| ,$$

where sup is taken over all points $(x, y) \in [a', a''] \cup [b', b'']$.

LEMMA 4.2.3. *If* δ *is sufficiently small* $(\omega(\delta) \leqslant C_2(\gamma))$, *then for any* $i > 0$

$$\left| \int\limits_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k (s) f_i^k (q_i^k (s))\, ds - \int\limits_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k (s) f_i^k (q_i^k (s))\, ds \right|$$

$$\leqslant C_3(\gamma) (\alpha\varepsilon + m\alpha\omega(\delta) + m\delta),$$

where the constants $C_2(\gamma)$, $C_3(\gamma)$ *depend only on* γ.

Proof. By (1), (2), (3) there exists a sufficiently small constant $C_2(\gamma)$ and a sufficiently large constant $C_3(\gamma)$ such that if $\omega(\delta) \leqslant C_2(\gamma)$ and for a point $a \in [a', a'']$ the inequalities $h_1([a', a]) \geqslant C_3(\gamma)\,\delta$; $h_1([a, a'']) \geqslant C_3(\gamma)\delta$ are satisfied, then for any $j \neq i$ $(j > 0)$ the level curve of the function q_j^k that passes through a intersects $[b', b'']$ of the level curve of q_i^k. Suppose that $\alpha > 2C_3(\gamma)\,\delta$ (if $\alpha \leqslant 2\,C_3(\gamma)\,\delta$, then the assertion of the lemma is trivial) and suppose that the segment $[\tilde{a}', \tilde{a}'']$ of the level curve of q_i^k is such that $[\tilde{a}', \tilde{a}''] \subset [a', a'']$ and $h_1([a', \tilde{a}']) = h_1([\tilde{a}'', a'']) = C_3(\gamma)\,\delta$. On the arc $[\tilde{a}', \tilde{a}'']$ we fix a system of points $a_1, a_2, \ldots, a_\nu$ $(\tilde{a}' = a_1,\ \tilde{a}'' = a_\nu)$, uniformly distributed along the length of this arc, and denote by b_r the point of intersection of $[b', b'']$ with the level curve of q_j^k that passes through a_r (here $j \neq i$ should for the time being be regarded as fixed). Using Lemma 4.2.1 we have

$$\left| \int\limits_{s \in [a', a'']} p_j^k (s) f_j^k (q_j^k (s))\, ds - \int\limits_{s \in [b', b'']} p_j^k (s) f_j^k (q_j^k (s))\, ds \right|$$

$$= \left| \int\limits_{s \in [a_1, a_\nu]} p_j^k (s) f_j^k (q_j^k (s))\, ds - \int\limits_{s \in [b_1, b_\nu]} p_j^k (s) f_j^k (q_j^k (s))\, ds \right|$$

$$+ O(\gamma)\, m\delta$$

$$= \lim_{\nu \to \infty} \Big| \sum_{r=1}^{\nu} p_j^k (a_r) f_j^k (q_j^k (a_r))\, h_1([a_r, a_{r+1}])$$

$$- \sum_{r=1}^{\nu} p_j^k (b_r) f_j^k (q_j^k (b_r))\, h_1([b_r, b_{r+1}]) \Big| + O(\gamma)\, m\delta$$

$$= \lim_{\nu \to \infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])$$

$$- \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])\,(1 + O(\gamma)\,\omega(\delta))$$

$$+ \sum_{r=1}^{\nu} (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}]) \Big| + O(\gamma)\, m\delta$$

$$= \lim_{\nu \to \infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])\, O(\gamma)\,\omega(\delta)$$

$$+ \sum_{r=1}^{\nu} f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}])\, O(\gamma)\,\omega(\delta) \Big| + O(\gamma)\, m\delta$$

$$= O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\delta = O(\gamma)\, m\,(\delta + \alpha\omega(\delta)).$$

Then

$$\Big| \int\limits_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds$$

$$\leqslant \Big| \int\limits_{s \in [a', a'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s \in [b', b'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds \Big|$$

$$+ \sum_{j \neq i} \Big| \int\limits_{s \in [a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds - \int\limits_{s \in [b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds \Big|$$

$$\leqslant C_4(\gamma)\, \alpha\varepsilon + n\,(\max_{j \neq i} m_j)\, C_5(\gamma)\, m\,(\delta + \alpha\omega(\delta))$$

$$\leqslant C_3(\gamma)\, (\alpha\varepsilon + m\delta + m\alpha\omega(\delta)).$$

This proves the lemma.

§ 3. *Deletion of dependent terms*

On a bounded closed set D we consider the space of linear superpositions of the form $\sum_{k=1}^{m} p_k(x, y) f_k(q(x, y))$, $(x, y) \in D$. Here the functions $\{p_k(x, y)\}$ and $q(x, y)$ are continuous and fixed, and $\{f_k(t)\}$ are arbitrary continuous functions of one variable. We assume that the function $q(x, y)$ is such that for any sequence $t_n \in q(D) \to t \in q(D)$ we have $\rho\,[e(q, t_n) \cap D, e(q, t) \cap D] \to 0$. We put

$$\lambda(t, D, q, p_1, \ldots, p_m) = \inf_{\{c_k\}} \sup_{(x, y) \in e(q, t) \cap D} \Big| \sum_{k=1}^{m} c_k p_k(x, y) \Big|,$$