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§ 2. *Estimate of the difference of the integrals of one term of a superposition along nearby level curves*

Let G be a region of the plane of the variables x and y , and $q_1(x, y)$ and $q_2(x, y)$ continuously differentiable functions satisfying in this region the following conditions: a) the partial derivatives with respect to x and with respect to y have modulus of continuity $\omega(\delta)$; b) the inequalities

$$0 < \gamma \leq | \text{grad} [q_i(x, y)] | \leq \frac{1}{\gamma} < \infty \quad (i = 1, 2)$$

are satisfied everywhere in G , where γ is a constant; c) for any point $(x, y) \in G$ the absolute value of the acute angle formed by the level curves of the functions $q_1(x, y)$ and $q_2(x, y)$ which pass through this point is greater than some positive constant γ .

LEMMA 4.2.1. *Let e'_{q_2} and e''_{q_2} be two level curves of the function q_2 and e'_{q_1} and e''_{q_1} level curves of the function q_1 ; $[a', a''] \subset G$ the segment of the curve e'_{q_1} with end-points $a' \in e'_{q_2}$ and $a'' \in e''_{q_2}$; $[b', b'']$ the segment of the curve e''_{q_1} with end-points $b' \in e'_{q_2}$ and $b'' \in e''_{q_2}$. Then*

$$h_1([b', b'']) \leq h_1([a', a'']) \times (1 + c_1(\gamma) \omega(\delta)),$$

where $\delta = d_1([a', a''] \cup [b', b''])$ and $c_1(\gamma)$ depends only on γ .

Proof. Since $q_2(a'') - q_2(a') = q_2(b'') - q_2(b')$, we have

$$\int_{s \in [a', a'']} \frac{\partial q_2}{\partial s} ds = \int_{s \in [b', b'']} \frac{\partial q_2}{\partial s} ds.$$

Consequently, $\frac{\partial q_2(a^*)}{\partial s} h_1([a', a'']) = \frac{\partial q_2(b^*)}{\partial s} h_1([b', b''])$, where $\frac{\partial q_2(a^*)}{\partial s}$

and $\frac{\partial q_2(b^*)}{\partial s}$ are the derivatives at the points $a^* \in [a', a'']$ and $b^* \in [b', b'']$

along the curves $[a', a'']$ and $[b', b'']$, respectively. We show that $\frac{\partial q_2(a^*)}{\partial s}$

$= \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta)$. We denote by q_2^* the derivative of q_2 at the point b^*

in the direction of $\tau(e'_{q_1}, a^*)$ and put $\alpha = \gamma \{ \tau[e''_{q_1}, b^*], \tau[e'_{q_1}, a^*] \}$. From

conditions a) and b) it follows that $\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1) \omega(\delta)$ and α

$= O(\gamma) \omega(\delta)$. We denote by β_1 and β_2 the values of the angles formed by the vectors $\tau [e''_{q_1}, b^*]$ and $\tau [e'_{q_1}, a^*]$ with the vector $\text{grad} [q_2(b^*)]$. We have

$$\left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| = |\text{grad} [q_2(b^*)]| |\cos \beta_2 - \cos \beta_1| = O(\gamma) \alpha = O(\gamma) \omega(\delta).$$

Thus,

$$\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1) \omega(\delta) = \frac{\partial q_2(b^*)}{\partial s} + O(1) \left\{ \left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| + \omega(\delta) \right\} = \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta).$$

Consequently,

$$\begin{aligned} h_1([b', b'']) &= h_1([a', a'']) \frac{\partial q_2(a^*)}{\partial s} \left(\frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \\ &= h_1([a', a'']) \left(1 + O(\gamma) \omega(\delta) \left(\frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \right) \\ &= h_1([a', a'']) (1 + O(\gamma) \omega(\gamma)), \end{aligned}$$

since by virtue of b) $\frac{\partial q_2(b^*)}{\partial s} > |\text{grad} [q_2(b^*)]| \sin \gamma$. This, proves the lemma.

LEMMA 4.2.2. Let $q_m(x, y)$ ($m=1, 2, \dots, N$) be continuously differentiable functions. In any region D we can find a subregion $G \subset D$, determine a constant $\gamma > 0$, and renumber the functions $\{q_m(x, y)\}$ with two indices so that the functions

$$q_i^k(x, y) = q_m(x, y) \quad (i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i; \sum_{i=0}^n m_i = N)$$

obtained after the renumbering satisfy the following conditions:

(1) when $i=0$, $q_i^k = \text{const}$ in G , and when $i > 0$, $\gamma \leq |\text{grad} [q_i^k(x, y)]| \leq \frac{1}{\gamma}$ for every point $(x, y) \in G$;

(2) the functions $q_i^k(x, y)$ ($i > 0$ fixed, $k=1, 2, \dots, m_i$) have in the region G identical sets of level curves, more precisely, in the region G , $q_i^k(x, y) \equiv \varphi_i^{k,l}(q_i^l(x, y))$, where $\varphi_i^{k,l}(t)$ is a strictly monotonic continuously differentiable function of t ;

(3) when $i \neq j$ ($i, j \neq 0$), then for any k and l the absolute value of the acute angle formed by the level curves of the functions $q_i^k(x, y)$ and $q_j^l(x, y)$ which pass through an arbitrary point $(x, y) \in G$ is greater than γ .

Proof. By the continuity of the partial derivatives of the functions $\{q_m(x, y)\}$ there exists a subregion $G^* \subset D$ inside which for any function $q_m(x, y)$ either $\text{grad } q_m(x, y) \equiv 0$ or $|\text{grad } q_m(x, y)|$ is greater than some positive constant. From the continuity of the partial derivatives of the functions $\{q_m(x, y)\}$ it follows also that there exists a subregion $G^{**} \subset G^*$ inside which for any pair of functions $q_r(x, y)$ and $q_s(x, y)$ one of two conditions holds: either $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$ in G^{**} , or for every point of G^{**} the level curves of $q_r(x, y)$ and $q_s(x, y)$ that pass through this point intersect at a non-zero angle ($D\left(\frac{q_r, q_s}{x, y}\right) \neq 0$ in G^{**}). From the implicit function theorem it follows that there exists a subregion $G \subset G^{**}$ in which condition (2) is satisfied for every pair of functions $q_r(x, y)$ and $q_s(x, y)$ with gradients different from zero and with determinant $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$.

We now renumber the functions $\{q_m(x, y)\}$ with two indices in such a way that only functions constant in G have lower index zero, and the same lower index is assigned to those functions whose level curves coincide identically in G . This proves the lemma.

We consider in the region G a superposition of the form $\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y))$, where $\{f_i^k(t)\}$ are continuous functions of one variable, $\{p_i^k(x, y)\}$ are continuous functions satisfying in G the condition $|p_i^k(x, y)| \leq \frac{1}{\gamma}$ and $\{q_i^k(x, y)\}$ are continuously differentiable functions satisfying in G conditions (1), (2), (3) of Lemma 4.2.2. Let $\omega(\delta)$ be the common modulus of continuity in G of the functions $\left\{p_i^k(x, y); \frac{\partial q_i^k(x, y)}{\partial x}; \frac{\partial q_i^k(x, y)}{\partial y}\right\}$. Let $[a', a'']$ and $[b', b'']$ be segments of the level curves of the functions $\{q_i^k(x, y)\}$ ($i > 0$ fixed) lying in G . Let

$$\alpha = h_1([a', a'']); \quad \delta = \rho([a', a''], [b', b'']);$$

$$\varepsilon = \sup \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right|;$$

$$m = \max_{i,k} \sup |f_i^k(q_i^k(x, y))|,$$

where sup is taken over all points $(x, y) \in [a', a''] \cup [b', b'']$.

LEMMA 4.2.3. *If δ is sufficiently small ($\omega(\delta) \leq C_2(\gamma)$), then for any $i > 0$*

$$\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \leq C_3(\gamma)(\alpha\varepsilon + m\alpha\omega(\delta) + m\delta),$$

where the constants $C_2(\gamma), C_3(\gamma)$ depend only on γ .

Proof. By (1), (2), (3) there exists a sufficiently small constant $C_2(\gamma)$ and a sufficiently large constant $C_3(\gamma)$ such that if $\omega(\delta) \leq C_2(\gamma)$ and for a point $a \in [a', a'']$ the inequalities $h_1([a', a]) \geq C_3(\gamma)\delta; h_1([a, a'']) \geq C_3(\gamma)\delta$ are satisfied, then for any $j \neq i$ ($j > 0$) the level curve of the function q_j^k that passes through a intersects $[b', b'']$ of the level curve of q_i^k . Suppose that $\alpha > 2C_3(\gamma)\delta$ (if $\alpha \leq 2C_3(\gamma)\delta$, then the assertion of the lemma is trivial)

and suppose that the segment $[\tilde{a}', \tilde{a}']$ of the level curve of q_i^k is such that $[\tilde{a}', \tilde{a}'] \subset [a', a'']$ and $h_1([a', \tilde{a}']) = h_1([\tilde{a}', a'']) = C_3(\gamma)\delta$. On the arc $[\tilde{a}', \tilde{a}']$ we fix a system of points a_1, a_2, \dots, a_ν ($\tilde{a}' = a_1, \tilde{a}'' = a_\nu$), uniformly distributed along the length of this arc, and denote by b_r the point of intersection of $[b', b'']$ with the level curve of q_j^k that passes through a_r (here $j \neq i$ should for the time being be regarded as fixed). Using Lemma 4.2.1 we have

$$\begin{aligned} & \left| \int_{s \in [a', a'']} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ &= \left| \int_{s \in [a_1, a_\nu]} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b_1, b_\nu]} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ &+ O(\gamma) m\delta \\ &= \lim_{\nu \rightarrow \infty} \left| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\ &\quad \left. - \sum_{r=1}^{\nu} p_j^k(b_r) f_j^k(q_j^k(b_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m\delta \end{aligned}$$

$$\begin{aligned}
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\
 &\quad - \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) (1 + O(\gamma) \omega(\delta)) \\
 &\quad \left. + \sum_{r=1}^v (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m \delta \\
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) O(\gamma) \omega(\delta) \right. \\
 &\quad \left. + \sum_{r=1}^v f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) O(\gamma) \omega(\delta) \right| + O(\gamma) m \delta \\
 &= O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \delta = O(\gamma) m (\delta + \alpha \omega(\delta)).
 \end{aligned}$$

Then

$$\begin{aligned}
 &\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right. \\
 &\leq \left| \int_{s \in [a', a'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\
 &\quad + \left| \sum_{j \neq i} \int_{s \in [a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\
 &\leq C_4(\gamma) \alpha \varepsilon + n (\max_{j \neq i} m_j) C_5(\gamma) m (\delta + \alpha \omega(\delta)) \\
 &\leq C_3(\gamma) (\alpha \varepsilon + m \delta + m \alpha \omega(\delta)).
 \end{aligned}$$

This proves the lemma.

§ 3. Deletion of dependent terms

On a bounded closed set D we consider the space of linear superpositions of the form $\sum_{k=1}^m p_k(x, y) f_k(q(x, y))$, $(x, y) \in D$. Here the functions $\{p_k(x, y)\}$ and $q(x, y)$ are continuous and fixed, and $\{f_k(t)\}$ are arbitrary continuous functions of one variable. We assume that the function $q(x, y)$ is such that for any sequence $t_n \in q(D) \rightarrow t \in q(D)$ we have $\rho[e(q, t_n) \cap D, e(q, t) \cap D] \rightarrow 0$. We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right|,$$