

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 23 (1977)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS  
**Autor:** Vitushkin, A. G.  
**Kapitel:** §2. Estimate of the difference of the integrals of one term of a superposition along nearby level curves  
**DOI:** <https://doi.org/10.5169/seals-48931>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 09.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

§ 2. Estimate of the difference of the integrals of one term  
of a superposition along nearby level curves

Let  $G$  be a region of the plane of the variables  $x$  and  $y$ , and  $q_1(x, y)$  and  $q_2(x, y)$  continuously differentiable functions satisfying in this region the following conditions: a) the partial derivatives with respect to  $x$  and with respect to  $y$  have modulus of continuity  $\omega(\delta)$ ; b) the inequalities

$$0 < \gamma \leqslant |\operatorname{grad} [q_i(x, y)]| \leqslant \frac{1}{\gamma} < \infty \quad (i = 1, 2)$$

are satisfied everywhere in  $G$ , where  $\gamma$  is a constant; c) for any point  $(x, y) \in G$  the absolute value of the acute angle formed by the level curves of the functions  $q_1(x, y)$  and  $q_2(x, y)$  which pass through this point is greater than some positive constant  $\gamma$ .

LEMMA 4.2.1. Let  $e'_{q_2}$  and  $e''_{q_2}$  be two level curves of the function  $q_2$  and  $e'_{q_1}$  and  $e''_{q_1}$  level curves of the function  $q_1$ ;  $[a', a''] \subset G$  the segment of the curve  $e'_{q_1}$  with end-points  $a' \in e'_{q_2}$  and  $a'' \in e''_{q_2}$ ;  $[b', b'']$  the segment of the curve  $e''_{q_1}$  with end-points  $b' \in e'_{q_2}$  and  $b'' \in e''_{q_2}$ . Then

$$h_1([b', b'']) \leqslant h_1([a', a'']) \times (1 + c_1(\gamma) \omega(\delta)),$$

where  $\delta = d_1([a', a''] \cup [b', b''])$  and  $c_1(\gamma)$  depends only on  $\gamma$ .

*Proof.* Since  $q_2(a'') - q_2(a') = q_2(b'') - q_2(b')$ , we have

$$\int_{s \in [a', a'']} \frac{\partial q_2}{\partial s} ds = \int_{s \in [b', b'']} \frac{\partial q_2}{\partial s} ds.$$

Consequently,  $\frac{\partial q_2(a^*)}{\partial s} h_1([a', a'']) = \frac{\partial q_2(b^*)}{\partial s} h_1([b', b''])$ , where  $\frac{\partial q_2(a^*)}{\partial s}$

and  $\frac{\partial q_2(b^*)}{\partial s}$  are the derivatives at the points  $a^* \in [a', a'']$  and  $b^* \in [b', b'']$

along the curves  $[a', a'']$  and  $[b', b'']$ , respectively. We show that  $\frac{\partial q_2(a^*)}{\partial s}$

$= \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta)$ . We denote by  $q_2^*$  the derivative of  $q_2$  at the point  $b^*$  in the direction of  $\tau(e'_{q_1}, a^*)$  and put  $\alpha = \gamma \{ \tau[e''_{q_1}, b^*], \tau[e'_{q_1}, a^*] \}$ . From conditions a) and b) it follows that  $\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1) \omega(\delta)$  and  $\alpha$

$= O(\gamma) \omega(\delta)$ . We denote by  $\beta_1$  and  $\beta_2$  the values of the angles formed by the vectors  $\tau [e_{q_1}^{\prime\prime}, b^*]$  and  $\tau [e_{q_1}', a^*]$  with the vector grad  $[q_2(b^*)]$ . We have

$$\begin{aligned} \left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| &= | \text{grad } [q_2(b^*)] | |\cos \beta_2 - \cos \beta_1| = O(\gamma) \alpha \\ &= O(\gamma) \omega(\delta). \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial q_2(a^*)}{\partial s} &= q_2^* + O(1) \omega(\delta) = \frac{\partial q_2(b^*)}{\partial s} \\ + O(1) \left\{ \left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| + \omega(\delta) \right\} &= \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta). \end{aligned}$$

Consequently,

$$\begin{aligned} h_1([b', b'']) &= h_1([a', a'']) \frac{\partial q_2(a^*)}{\partial s} \left( \frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \\ &= h_1([a', a'']) \left( 1 + O(\gamma) \omega(\delta) \left( \frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \right) \\ &= h_1([a', a'']) (1 + O(\gamma) \omega(\gamma)), \end{aligned}$$

since by virtue of b)  $\frac{\partial q_2(b^*)}{\partial s} > | \text{grad } [q_2(b^*)] | \sin \gamma$ . This, proves the lemma.

LEMMA 4.2.2. Let  $q_m(x, y)$  ( $m=1, 2, \dots, N$ ) be continuously differentiable functions. In any region  $D$  we can find a subregion  $G \subset D$ , determine a constant  $\gamma > 0$ , and renumber the functions  $\{q_m(x, y)\}$  with two indices so that the functions

$$q_i^k(x, y) = q_m(x, y) \quad (i = 0, 1, 2, \dots, n; k = 1, 2, \dots, m_i; \sum_{i=0}^n m_i = N)$$

obtained after the renumbering satisfy the following conditions:

(1) when  $i = 0$ ,  $q_i^k = \text{const}$  in  $G$ , and when  $i > 0$ ,  $\gamma \leq | \text{grad } [q_i^k(x, y)] | \leq \frac{1}{\gamma}$  for every point  $(x, y) \in G$ ;

(2) the functions  $q_i^k(x, y)$  ( $i > 0$  fixed,  $k = 1, 2, \dots, m_i$ ) have in the region  $G$  identical sets of level curves, more precisely, in the region  $G$ ,  $q_i^k(x, y) \equiv \varphi_i^{k,l}(q_i^l(x, y))$ , where  $\varphi_i^{k,l}(t)$  is a strictly monotonic continuously differentiable function of  $t$ ;

(3) when  $i \neq j$  ( $i, j \neq 0$ ), then for any  $k$  and  $l$  the absolute value of the acute angle formed by the level curves of the functions  $q_i^k(x, y)$  and  $q_j^l(x, y)$  which pass through an arbitrary point  $(x, y) \in G$  is greater than  $\gamma$ .

*Proof.* By the continuity of the partial derivatives of the functions  $\{q_m(x, y)\}$  there exists a subregion  $G^* \subset D$  inside which for any function  $q_m(x, y)$  either  $\text{grad } q_m(x, y) \equiv 0$  or  $|\text{grad } q_m(x, y)|$  is greater than some positive constant. From the continuity of the partial derivatives of the functions  $\{q_m(x, y)\}$  it follows also that there exists a subregion  $G^{**} \subset G^*$  inside which for any pair of functions  $q_r(x, y)$  and  $q_s(x, y)$  one of two conditions holds: either  $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$  in  $G^{**}$ , or for every point of  $G^{**}$  the level curves of  $q_r(x, y)$  and  $q_s(x, y)$  that pass through this point intersect at a non-zero angle ( $D\left(\frac{q_r, q_s}{x, y}\right) \neq 0$  in  $G^{**}$ ). From the implicit function theorem it follows that there exists a subregion  $G \subset G^{**}$  in which condition (2) is satisfied for every pair of functions  $q_r(x, y)$  and  $q_s(x, y)$  with gradients different from zero and with determinant  $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$ .

We now renumber the functions  $\{q_m(x, y)\}$  with two indices in such a way that only functions constant in  $G$  have lower index zero, and the same lower index is assigned to those functions whose level curves coincide identically in  $G$ . This proves the lemma.

We consider in the region  $G$  a superposition of the form  $\sum_{i=0}^n \sum_{k=1}^{m_i} p_r(x, y) f_i^k(q_i^k(x, y))$ , where  $\{f_i^k(t)\}$  are continuous functions of one variable,  $\{p_i^k(x, y)\}$  are continuous functions satisfying in  $G$  the condition  $|p_i^k(x, y)| \leq \frac{1}{\gamma}$  and  $\{q_i^k(x, y)\}$  are continuously differentiable functions satisfying in  $G$  conditions (1), (2), (3) of Lemma 4.2.2. Let  $\omega(\delta)$  be the common modulus of continuity in  $G$  of the functions  $\left\{p_i^k(x, y); \frac{\partial q_i^k(x, y)}{\partial x}; \frac{\partial q_i^k(x, y)}{\partial y}\right\}$ . Let  $[a', a'']$  and  $[b', b'']$  be segments of the level curves of the functions  $\{q_i^k(x, y)\}$  ( $i > 0$  fixed) lying in  $G$ . Let

$$\alpha = h_1([a', a'']); \quad \delta = \rho([a', a''], [b', b'']);$$

$$\varepsilon = \sup \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right|;$$

$$m = \max_{i,k} \sup |f_i^k(q_i^k(x, y))|,$$

where sup is taken over all points  $(x, y) \in [a', a''] \cup [b', b'']$ .

LEMMA 4.2.3. If  $\delta$  is sufficiently small ( $\omega(\delta) \leq C_2(\gamma)$ ), then for any  $i > 0$

$$\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\ \leq C_3(\gamma) (\alpha \varepsilon + m \alpha \omega(\delta) + m \delta),$$

where the constants  $C_2(\gamma), C_3(\gamma)$  depend only on  $\gamma$ .

*Proof.* By (1), (2), (3) there exists a sufficiently small constant  $C_2(\gamma)$  and a sufficiently large constant  $C_3(\gamma)$  such that if  $\omega(\delta) \leq C_2(\gamma)$  and for a point  $a \in [a', a'']$  the inequalities  $h_1([a', a]) \geq C_3(\gamma) \delta; h_1([a, a'']) \geq C_3(\gamma) \delta$  are satisfied, then for any  $j \neq i$  ( $j > 0$ ) the level curve of the function  $q_j^k$  that passes through  $a$  intersects  $[b', b'']$  of the level curve of  $q_i^k$ . Suppose that  $\alpha > 2C_3(\gamma) \delta$  (if  $\alpha \leq 2C_3(\gamma) \delta$ , then the assertion of the lemma is trivial) and suppose that the segment  $[\tilde{a}', \tilde{a}'']$  of the level curve of  $q_i^k$  is such that  $[\tilde{a}', \tilde{a}''] \subset [a', a'']$  and  $h_1([a', \tilde{a}']) = h_1([\tilde{a}'', a'']) = C_3(\gamma) \delta$ . On the arc  $[\tilde{a}', \tilde{a}'']$  we fix a system of points  $a_1, a_2, \dots, a_v$  ( $\tilde{a}' = a_1, \tilde{a}'' = a_v$ ), uniformly distributed along the length of this arc, and denote by  $b_r$  the point of intersection of  $[b', b'']$  with the level curve of  $q_j^k$  that passes through  $a_r$  (here  $j \neq i$  should for the time being be regarded as fixed). Using Lemma 4.2.1 we have

$$\left| \int_{s \in [a', a'']} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ = \left| \int_{s \in [a_1, a_v]} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b_1, b_v]} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ + O(\gamma) m \delta \\ = \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\ \left. - \sum_{r=1}^v p_j^k(b_r) f_j^k(q_j^k(b_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m \delta$$

$$\begin{aligned}
&= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\
&\quad - \left. \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) (1 + O(\gamma) \omega(\delta)) \right. \\
&\quad + \left. \sum_{r=1}^v (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m \delta \\
&= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) O(\gamma) \omega(\delta) \right. \\
&\quad + \left. \sum_{r=1}^v f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) O(\gamma) \omega(\delta) \right| + O(\gamma) m \delta \\
&= O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \delta = O(\gamma) m (\delta + \alpha \omega(\delta)).
\end{aligned}$$

Then

$$\begin{aligned}
&\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\
&\leqslant \left| \int_{s \in [a', a'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\
&\quad + \sum_{j \neq i} \left| \int_{s \in [a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\
&\leqslant C_4(\gamma) \alpha \varepsilon + n \left( \max_{j \neq i} m_j \right) C_5(\gamma) m (\delta + \alpha \omega(\delta)) \\
&\leqslant C_3(\gamma) (\alpha \varepsilon + m \delta + m \alpha \omega(\delta)).
\end{aligned}$$

This proves the lemma.

### § 3. *Deletion of dependent terms*

On a bounded closed set  $D$  we consider the space of linear superpositions of the form  $\sum_{k=1}^m p_k(x, y) f_k(q(x, y))$ ,  $(x, y) \in D$ . Here the functions  $\{p_k(x, y)\}$  and  $q(x, y)$  are continuous and fixed, and  $\{f_k(t)\}$  are arbitrary continuous functions of one variable. We assume that the function  $q(x, y)$  is such that for any sequence  $t_n \in q(D) \rightarrow t \in q(D)$  we have  $\rho[e(q, t_n) \cap D, e(q, t) \cap D] \rightarrow 0$ . We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right|,$$