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CHAPTER 4. — LINEAR SUPERPOSITIONS

In this chapter we prove that there exist analytic functions which are not representable by means of linear superpositions of smooth functions of one variable.

§ 1. *Notation*

Throughout we assume that all the functions are defined and continuous for all values of the arguments. If we say that a function is continuously differentiable, we mean by this that its first partial derivatives are defined and continuous for all values of the arguments; $z = (x, y)$ is the point of the plane with coordinates x and y ; $\text{grad } [q(z)]$ is the gradient of the function $q(z)$, that is, the vector-function with coordinates $\frac{\partial q}{\partial x}$ and $\frac{\partial q}{\partial y}$;

$$D \left(\begin{array}{c} q_1, q_2 \\ x, y \end{array} \right) = \begin{vmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} \end{vmatrix}$$

is the Jacobian of the pair of functions q_1 and q_2 .

$q(D)$ is the image of the set D under the mapping effected by the function $q(x, y)$; $q^{-1}(\delta)$ is the complete inverse image of the interval δ on the axis of values of the function $q(x, y)$.

$e(q, t)$ is the set of level t of the function $q = q(x, y)$.

$\tau(e, z)$ is the unit tangent vector to the curve e at the point $z \in e$.

$\gamma(\tau_1, \tau_2)$ is the absolute value of the acute angle between the vectors τ_1 and τ_2 .

$h_1(e)$ is the length of the set e .

$d_1(e)$ is the one-dimensional diameter of the set e .

$O(\gamma)$ is a quantity bounded by a constant depending only on γ .

$\rho(A_1, A_2)$ is the distance between the sets A_1 and A_2 in the sense of deviation, more precisely

$$\rho(A_1, A_2) = \max \left\{ \sup_{z_1 \in A_1} \inf_{z_2 \in A_2} \rho(z_1, z_2), \sup_{z_2 \in A_2} \inf_{z_1 \in A_1} \rho(z_1, z_2) \right\},$$

where $\rho(z_1, z_2)$ is the distance between the points z_1 and z_2 .

§ 2. *Estimate of the difference of the integrals of one term of a superposition along nearby level curves*

Let G be a region of the plane of the variables x and y , and $q_1(x, y)$ and $q_2(x, y)$ continuously differentiable functions satisfying in this region the following conditions: a) the partial derivatives with respect to x and with respect to y have modulus of continuity $\omega(\delta)$; b) the inequalities

$$0 < \gamma \leq | \text{grad} [q_i(x, y)] | \leq \frac{1}{\gamma} < \infty \quad (i = 1, 2)$$

are satisfied everywhere in G , where γ is a constant; c) for any point $(x, y) \in G$ the absolute value of the acute angle formed by the level curves of the functions $q_1(x, y)$ and $q_2(x, y)$ which pass through this point is greater than some positive constant γ .

LEMMA 4.2.1. *Let e'_{q_2} and e''_{q_2} be two level curves of the function q_2 and e'_{q_1} and e''_{q_1} level curves of the function q_1 ; $[a', a''] \subset G$ the segment of the curve e'_{q_1} with end-points $a' \in e'_{q_2}$ and $a'' \in e''_{q_2}$; $[b', b'']$ the segment of the curve e''_{q_1} with end-points $b' \in e'_{q_2}$ and $b'' \in e''_{q_2}$. Then*

$$h_1([b', b'']) \leq h_1([a', a'']) \times (1 + c_1(\gamma) \omega(\delta)),$$

where $\delta = d_1([a', a''] \cup [b', b''])$ and $c_1(\gamma)$ depends only on γ .

Proof. Since $q_2(a'') - q_2(a') = q_2(b'') - q_2(b')$, we have

$$\int_{s \in [a', a'']} \frac{\partial q_2}{\partial s} ds = \int_{s \in [b', b'']} \frac{\partial q_2}{\partial s} ds.$$

Consequently, $\frac{\partial q_2(a^*)}{\partial s} h_1([a', a'']) = \frac{\partial q_2(b^*)}{\partial s} h_1([b', b''])$, where $\frac{\partial q_2(a^*)}{\partial s}$

and $\frac{\partial q_2(b^*)}{\partial s}$ are the derivatives at the points $a^* \in [a', a'']$ and $b^* \in [b', b'']$

along the curves $[a', a'']$ and $[b', b'']$, respectively. We show that $\frac{\partial q_2(a^*)}{\partial s}$

$= \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta)$. We denote by q_2^* the derivative of q_2 at the point b^*

in the direction of $\tau(e'_{q_1}, a^*)$ and put $\alpha = \gamma \{ \tau[e''_{q_1}, b^*], \tau[e'_{q_1}, a^*] \}$. From

conditions a) and b) it follows that $\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1) \omega(\delta)$ and α

$= O(\gamma) \omega(\delta)$. We denote by β_1 and β_2 the values of the angles formed by the vectors $\tau [e''_{q_1}, b^*]$ and $\tau [e'_{q_1}, a^*]$ with the vector $\text{grad} [q_2(b^*)]$. We have

$$\left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| = |\text{grad} [q_2(b^*)]| |\cos \beta_2 - \cos \beta_1| = O(\gamma) \alpha = O(\gamma) \omega(\delta).$$

Thus,

$$\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1) \omega(\delta) = \frac{\partial q_2(b^*)}{\partial s} + O(1) \left\{ \left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| + \omega(\delta) \right\} = \frac{\partial q_2(b^*)}{\partial s} + O(\gamma) \omega(\delta).$$

Consequently,

$$\begin{aligned} h_1([b', b'']) &= h_1([a', a'']) \frac{\partial q_2(a^*)}{\partial s} \left(\frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \\ &= h_1([a', a'']) \left(1 + O(\gamma) \omega(\delta) \left(\frac{\partial q_2(b^*)}{\partial s} \right)^{-1} \right) \\ &= h_1([a', a'']) (1 + O(\gamma) \omega(\gamma)), \end{aligned}$$

since by virtue of b) $\frac{\partial q_2(b^*)}{\partial s} > |\text{grad} [q_2(b^*)]| \sin \gamma$. This, proves the lemma.

LEMMA 4.2.2. Let $q_m(x, y)$ ($m=1, 2, \dots, N$) be continuously differentiable functions. In any region D we can find a subregion $G \subset D$, determine a constant $\gamma > 0$, and renumber the functions $\{q_m(x, y)\}$ with two indices so that the functions

$$q_i^k(x, y) = q_m(x, y) \quad (i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i; \sum_{i=0}^n m_i = N)$$

obtained after the renumbering satisfy the following conditions:

(1) when $i=0$, $q_i^k = \text{const}$ in G , and when $i > 0$, $\gamma \leq |\text{grad} [q_i^k(x, y)]| \leq \frac{1}{\gamma}$ for every point $(x, y) \in G$;

(2) the functions $q_i^k(x, y)$ ($i > 0$ fixed, $k=1, 2, \dots, m_i$) have in the region G identical sets of level curves, more precisely, in the region G , $q_i^k(x, y) \equiv \varphi_i^{k,l}(q_i^l(x, y))$, where $\varphi_i^{k,l}(t)$ is a strictly monotonic continuously differentiable function of t ;

(3) when $i \neq j$ ($i, j \neq 0$), then for any k and l the absolute value of the acute angle formed by the level curves of the functions $q_i^k(x, y)$ and $q_j^l(x, y)$ which pass through an arbitrary point $(x, y) \in G$ is greater than γ .

Proof. By the continuity of the partial derivatives of the functions $\{q_m(x, y)\}$ there exists a subregion $G^* \subset D$ inside which for any function $q_m(x, y)$ either $\text{grad } q_m(x, y) \equiv 0$ or $|\text{grad } q_m(x, y)|$ is greater than some positive constant. From the continuity of the partial derivatives of the functions $\{q_m(x, y)\}$ it follows also that there exists a subregion $G^{**} \subset G^*$ inside which for any pair of functions $q_r(x, y)$ and $q_s(x, y)$ one of two conditions holds: either $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$ in G^{**} , or for every point of G^{**} the level curves of $q_r(x, y)$ and $q_s(x, y)$ that pass through this point intersect at a non-zero angle ($D\left(\frac{q_r, q_s}{x, y}\right) \neq 0$ in G^{**}). From the implicit function theorem it follows that there exists a subregion $G \subset G^{**}$ in which condition (2) is satisfied for every pair of functions $q_r(x, y)$ and $q_s(x, y)$ with gradients different from zero and with determinant $D\left(\frac{q_r, q_s}{x, y}\right) \equiv 0$.

We now renumber the functions $\{q_m(x, y)\}$ with two indices in such a way that only functions constant in G have lower index zero, and the same lower index is assigned to those functions whose level curves coincide identically in G . This proves the lemma.

We consider in the region G a superposition of the form $\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y))$, where $\{f_i^k(t)\}$ are continuous functions of one variable, $\{p_i^k(x, y)\}$ are continuous functions satisfying in G the condition $|p_i^k(x, y)| \leq \frac{1}{\gamma}$ and $\{q_i^k(x, y)\}$ are continuously differentiable functions satisfying in G conditions (1), (2), (3) of Lemma 4.2.2. Let $\omega(\delta)$ be the common modulus of continuity in G of the functions $\left\{p_i^k(x, y); \frac{\partial q_i^k(x, y)}{\partial x}; \frac{\partial q_i^k(x, y)}{\partial y}\right\}$. Let $[a', a'']$ and $[b', b'']$ be segments of the level curves of the functions $\{q_i^k(x, y)\}$ ($i > 0$ fixed) lying in G . Let

$$\alpha = h_1([a', a'']); \quad \delta = \rho([a', a''], [b', b'']);$$

$$\varepsilon = \sup \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right|;$$

$$m = \max_{i,k} \sup |f_i^k(q_i^k(x, y))|,$$

where sup is taken over all points $(x, y) \in [a', a''] \cup [b', b'']$.

LEMMA 4.2.3. *If δ is sufficiently small ($\omega(\delta) \leq C_2(\gamma)$), then for any $i > 0$*

$$\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \leq C_3(\gamma)(\alpha\varepsilon + m\alpha\omega(\delta) + m\delta),$$

where the constants $C_2(\gamma), C_3(\gamma)$ depend only on γ .

Proof. By (1), (2), (3) there exists a sufficiently small constant $C_2(\gamma)$ and a sufficiently large constant $C_3(\gamma)$ such that if $\omega(\delta) \leq C_2(\gamma)$ and for a point $a \in [a', a'']$ the inequalities $h_1([a', a]) \geq C_3(\gamma)\delta; h_1([a, a'']) \geq C_3(\gamma)\delta$ are satisfied, then for any $j \neq i$ ($j > 0$) the level curve of the function q_j^k that passes through a intersects $[b', b'']$ of the level curve of q_i^k . Suppose that $\alpha > 2C_3(\gamma)\delta$ (if $\alpha \leq 2C_3(\gamma)\delta$, then the assertion of the lemma is trivial)

and suppose that the segment $[\tilde{a}', \tilde{a}']$ of the level curve of q_i^k is such that $[\tilde{a}', \tilde{a}'] \subset [a', a'']$ and $h_1([a', \tilde{a}']) = h_1([\tilde{a}', a'']) = C_3(\gamma)\delta$. On the arc $[\tilde{a}', \tilde{a}']$ we fix a system of points a_1, a_2, \dots, a_ν ($\tilde{a}' = a_1, \tilde{a}'' = a_\nu$), uniformly distributed along the length of this arc, and denote by b_r the point of intersection of $[b', b'']$ with the level curve of q_j^k that passes through a_r (here $j \neq i$ should for the time being be regarded as fixed). Using Lemma 4.2.1 we have

$$\begin{aligned} & \left| \int_{s \in [a', a'']} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ &= \left| \int_{s \in [a_1, a_\nu]} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b_1, b_\nu]} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\ &+ O(\gamma) m\delta \\ &= \lim_{\nu \rightarrow \infty} \left| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\ &\quad \left. - \sum_{r=1}^{\nu} p_j^k(b_r) f_j^k(q_j^k(b_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m\delta \end{aligned}$$

$$\begin{aligned}
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) \right. \\
 &\quad - \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) (1 + O(\gamma) \omega(\delta)) \\
 &\quad \left. + \sum_{r=1}^v (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) \right| + O(\gamma) m \delta \\
 &= \lim_{v \rightarrow \infty} \left| \sum_{r=1}^v p_j^k(a_r) f_j^k(q_j^k(a_r)) h_1([a_r, a_{r+1}]) O(\gamma) \omega(\delta) \right. \\
 &\quad \left. + \sum_{r=1}^v f_j^k(q_j^k(a_r)) h_1([b_r, b_{r+1}]) O(\gamma) \omega(\delta) \right| + O(\gamma) m \delta \\
 &= O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \alpha \omega(\delta) + O(\gamma) m \delta = O(\gamma) m (\delta + \alpha \omega(\delta)).
 \end{aligned}$$

Then

$$\begin{aligned}
 &\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right. \\
 &\leq \left| \int_{s \in [a', a'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds - \int_{s \in [b', b'']} \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s)) ds \right| \\
 &\quad + \left| \sum_{j \neq i} \int_{s \in [a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s)) ds \right| \\
 &\leq C_4(\gamma) \alpha \varepsilon + n (\max_{j \neq i} m_j) C_5(\gamma) m (\delta + \alpha \omega(\delta)) \\
 &\leq C_3(\gamma) (\alpha \varepsilon + m \delta + m \alpha \omega(\delta)).
 \end{aligned}$$

This proves the lemma.

§ 3. Deletion of dependent terms

On a bounded closed set D we consider the space of linear superpositions of the form $\sum_{k=1}^m p_k(x, y) f_k(q(x, y))$, $(x, y) \in D$. Here the functions $\{p_k(x, y)\}$ and $q(x, y)$ are continuous and fixed, and $\{f_k(t)\}$ are arbitrary continuous functions of one variable. We assume that the function $q(x, y)$ is such that for any sequence $t_n \in q(D) \rightarrow t \in q(D)$ we have $\rho[e(q, t_n) \cap D, e(q, t) \cap D] \rightarrow 0$. We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right|,$$

where inf is taken over all sets of numbers $\{c_k\}$ for which $\max_k |c_k| = 1$. The function $\lambda(t, D, q, \{p_k\})$, as a function of t , is defined only on the set $q(D)$.

LEMMA 4.3.1. *The function $\lambda(t, D, q, \{p_k\})$ depends continuously on t .*

Proof. The linear combinations $\sum_{k=1}^m c_k p_k(x, y)$ for all possible systems of numbers $\{c_k\}$ for which $\max_k |c_k| \leq 1$, form an equicontinuous set of functions, considered on the bounded closed set D . Consequently, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $|t_1 - t_2| < \delta$, then

$$\left| \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k p_k(x, y) \right| - \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k p_k(x, y) \right| \right| < \varepsilon$$

simultaneously for all systems of numbers $\{c_k\}$ such that $\max_k |c_k| \leq 1$.

For definiteness, suppose that $\lambda(t_2, D, q, \{p_k\}) \geq \lambda(t_1, D, q, \{p_k\})$.

Since the expression $\sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k p_k(x, y) \right|$ depends continuously on the coefficients $\{c_k\}$, there exists a system of numbers $\{c_k^1\}$ such that $\max_k |c_k^1| = 1$ and

$$\lambda(t_1, D, q, \{p_k\}) = \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|.$$

Since

$$\lambda(t_2, D, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|,$$

we have

$$0 \leq \lambda(t_2) - \lambda(t_1) \leq \sup_{(x, y) \in e(q, t_2)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right|$$

$$- \sup_{(x, y) \in e(q, t_1)} \left| \sum_{k=1}^m c_k^1 p_k(x, y) \right| < \varepsilon.$$

This proves the lemma.

LEMMA 4.3.2. *The function $\lambda(t, D, q, \{p_k\})$ depends continuously on D in the sense that there exists a function $\mu(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, having the property: if the set $D_\varepsilon \subset D$ is such that, for any t , $D_\varepsilon \cap e(q, t)$ forms an ε -net in the set $e(q, t) \cap D$, then*

$$\max_{t \in q(D)} \left| \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \right| \leq \mu(\varepsilon).$$

Proof. Using the equicontinuity of the set of functions $\sum_{k=1}^n c_k p_k(x, y)$ where $\max_k |c_k| \leq 1$, we conclude that there exists a function $\mu(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ such that the inequality

$$0 \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k p_k(x, y) \right| - \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k p_k(x, y) \right| \leq \mu(\varepsilon).$$

uniformly over all $t \in q(D)$ and over all systems of numbers $\{c_k\}$ for which $\max_k |c_k| \leq 1$. For any $\varepsilon > 0$ there exists a system of numbers $\{c_k^\varepsilon\}$ such that $\max_k |c_k^\varepsilon| = 1$ and

$$\lambda(t, D_\varepsilon, q, \{p_k\}) = \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right|.$$

Since for any ε

$$\lambda(t, D, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right|$$

and, on the other hand, $\lambda(t, D, q, \{p_k\}) \geq \lambda(t, D_\varepsilon, q, \{p_k\})$ (we recall that $D_\varepsilon \subset D$), we have

$$0 \leq \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \leq \sup_{(x, y) \in e(q, t) \cap D} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right| - \sup_{(x, y) \in e(q, t) \cap D_\varepsilon} \left| \sum_{k=1}^m c_k^\varepsilon p_k(x, y) \right| < \mu(\varepsilon).$$

This proves the lemma.

LEMMA 4.3.3. *Let F be a closed set on the t -axis; $F \subset q(D)$. For every $t \in F$, suppose that there exists one and only one system of numbers $\{C_k\}$ ($\max_k |C_k| = 1$) such that $\sum_{k=1}^m C_k p_k(x, y) \equiv 0$ on the set $e(q, t) \cap D$. Then each of the functions $\{C_k(t)\}$ depends continuously on t on the set F .*

Proof. Suppose that $t_n \in F, t \in F$ and $t_n \rightarrow t$. We put $\overline{\lim}_{n \rightarrow \infty} C_k(t_n) = \tilde{C}_k$ and $\underline{\lim}_{n \rightarrow \infty} C_k(t_n) = \tilde{\tilde{C}}_k$. Since $\sum_{k=1}^m C_k(t_n) p_k(x, y) \equiv 0$ on the set $e(q, t_n) \cap D$ and $\rho[e(q, t) \cap D, e(q, t_n) \cap D] \rightarrow 0$ as $n \rightarrow \infty$, we have $\sum_{k=1}^m \tilde{C}_k p_k(x, y)$

$\equiv 0 \equiv \sum_{k=1}^m \tilde{C}_k p_k(x, y)$ on the set $e(q, t) \cap D$. Consequently, by the condition of the lemma, $\tilde{C}_k = \tilde{C}_k = C_k(t)$. This proves the lemma.

LEMMA 4.3.4. *Suppose that $\lambda(t, D, q, \{p_k\}) \equiv 0$ on some non-empty portion δ of the set $q(D)$. Then there is a non-empty portion $\delta^* \subset \delta$ and an index l such that for any continuous functions $\{f_k(t)\}$ there are continuous functions $\{f_k^*(t)\}$ such that*

$$\sum_{k \neq l} f_k^*(q(x, y)) p_k(x, y) = \sum_{k=1}^m f_k(q(x, y)) p_k(x, y)$$

on the set $q^{-1}(\delta^*) \cap D$.

We recall that a portion δ of a set E is that part of it which lies in the interval δ .

Proof. We prove the lemma by induction on m . For $m = 1$ the assertion of the lemma is obvious. We denote by δ_k the set of all points t of the portion δ for which $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$. By Lemma 4.3.1, the set is closed. Two cases are possible.

1) For some k the set δ_k contains a non-empty portion δ'_k of the set $q(D)$. Since $\lambda(t, D, q, p_1, \dots, p_{k-1}, p_{k+1}, \dots, p_m) = 0$ for every $t \in \delta'_k$, then by the inductive hypothesis there is a non-empty portion $\delta^* \subset \delta'_k$ and an index $l \neq k$ such that for any continuous functions $f_1(t), \dots, f_{k-1}(t), f_{k+1}(t), \dots, f_m(t)$ there are continuous functions $f_1^*(t), \dots, f_{k-1}^*(t), f_{k+1}^*(t), \dots, f_m^*(t)$ such that

$$\sum_{i \neq k} f_i(q(x, y)) p_i(x, y) = \sum_{i \neq k, l} f_i^*(q(x, y)) p_i(x, y).$$

on the set $q^{-1}(\delta^*) \cap D$. Putting $f_k^*(t) = f_k(t)$, we obtain

$$\sum_{i=1}^m f_i(q(x, y)) p_i(x, y) = \sum_{i \neq l} f_i^*(q(x, y)) p_i(x, y).$$

So in case 1) the lemma is proved.

2) None of the sets δ_k contains non-empty portions of the set $q(D)$, that is, $\bigcup_{k=1}^m \delta_k$ is nowhere dense in $q(D)$. Therefore there exists a non-empty portion $\delta^* \subset \delta \setminus \bigcup_{k=1}^m \delta_k$. Since $\lambda(t, D, q, \{p_k\}) \equiv 0$ on δ^* , for every $t \in \delta^*$ there are numbers $\{C_k(t)\}$ ($\max_k |C_k(t)| = 1$) such that $\sum_{k=1}^m C_k$

$(q(x, y)) p_k(x, y) \equiv 0$ on $e(q, t) \cap D$. If we had $C_k(t) = 0$ for some k , then it would turn out that $t \in \delta_k$. Consequently, $C_k(t) \neq 0$ for any k . We show that for every $t \in \delta^*$ the numbers $\{C_k(t)\}$ are uniquely determined. Assume the contrary. Then there are numbers $\{C'_k(t)\}$ ($\max |C'_k(t)| = 1$) such that $\sum_{k=1}^m C'_k(q(x, y)) p_k(x, y) = 0$ on $e(q, t) \cap D$ and $C_k \neq C'_k$ for some k . Then

$$\sum_{k \neq 1} [C_k(t) C'_1(t) - C'_k(t) C_1(t)] p_k(x, y) = \sum_{k \neq 1} C'_k(t) p_k(x, y) \equiv 0$$

on $e(q, t) \cap D$ and in addition, $C''_k \neq 0$ for some k . Consequently, $t \in \delta_1$. So we have obtained a contradiction, and the uniqueness of the choice of the numbers $C_k(t)$ is proved. Further, we may regard $\{C_k(t)\}$ as single-valued functions of t on the portion δ^* . By Lemma 4.3.3, the functions $C_k(t)$ are continuous and, as noted above, $C_k(t) \neq 0$ for any $t \in \delta^*$. Then

$$p_1(x, y) = \sum_{k=2}^m -\frac{C_k(q(x, y))}{C_1(q(x, y))} p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D.$$

Putting $f(t) = f_k(t) - \frac{C_k(t)}{C_1(t)} f_1(t)$, $t \in \delta^*$, we have $\sum_{k=2}^m f_k^*(q(x, y)) p_k(x, y)$

$$\begin{aligned} &= \sum_{k=1}^m f_k(q) p_k(x, y) - \sum_{k=2}^m \frac{C_k(q)}{C_1(q)} p_k(x, y) \\ &= \sum_{k=2}^m f_k(q) p_k(x, y) + f_1(q) p_1(x, y) \\ &= \sum_{k=1}^m f_k(q(x, y)) p_k(x, y), \quad (x, y) \in q^{-1}(\delta^*) \cap D. \end{aligned}$$

This proves the lemma.

§ 4. *Reduction of linear superpositions to a form with independent terms*

We fix the continuous functions $p_i^k(x, y)$ and continuously differentiable functions $q_i(x, y)$ ($i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i$) $n \geq 2$, where $\{q_i(x, y)\}$ satisfy in D conditions (1) and (3) of Lemma 4.2.2, and we consider in D superpositions of the form

$$\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y)),$$

where $\{f_i^k(t)\}$ are arbitrary continuous functions of one variable.

We call a bounded closed region $G \subset D$ polyhedral if the boundary of G consists of a finite number of mutually non-intersecting simple closed contours that are unions of a finite number of segments of level curves of the functions $q_i(x, y)$ ($i = 1, 2, \dots, n$). Let $G \subset D$ be a polyhedral region. We denote by Γ_i the set of those $t \in q_i(G)$ for which the set $e(q_i, t) \cap G$ contains a segment of a level curve belonging to the boundary of G . For any i the set Γ_i consists of a finite number of points. By property (1) of the functions $\{q_i(x, y)\}$ for every i and for all points $t_0 \in q_i(G) \setminus \Gamma_i$ there exists $\lim_{t \rightarrow t_0} e(q_i, t) \Rightarrow e(q_0, t_0)$. If $t_0 \in \Gamma_i$, then the last assertion need not hold, but in any case there exists $\lim_{t \rightarrow +t_0} e(q_i, t) \subset e(q_i, t_0)$ and $\lim_{t \rightarrow -t_0} e(q_i, t) \supset e(q_i, t_0)$ where the limit is taken over the points $t \in q_i(G)$. Here the limit is understood in the sense of the distance $\rho(e(q_i, t), e(q_i, t_0))$.

LEMMA 4.4.1. *There is a region $G \subset D$ and a system of numbers $\tau_i^k = 0$ or 1 ($i = 0, 1, 2, \dots, n; k = 1, 2, \dots, m_i$) such that*

(4) *for any i and for any continuous functions $\{\varphi_i^k(t)\}$ there exist continuous functions $\{f_i^k(t)\}$ such that in G*

$$\sum_{k=1}^{m_i} p_i^k(x, y) \varphi_i^k(q_i(x, y)) \equiv \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y));$$

(5*) *for any polyhedral region $G^* \subset G$ and any i , the set*

$$\{t : \lambda(t, G^*, q_i, p_i^{k_1}, \dots, p_i^{k_s}) = 0\}$$

is nowhere dense in $q_i(G^)$, where*

$$k_1 = k_1(i), k_2 = k_2(i), \dots, k_s = k_s(i)$$

is the set of all values of k for which $\tau_i^k = 1$.

Proof. If $i = 0$, then by (1) the set $q_0(D)$ consists of only one point. We choose a region $G_0 \subset D$ and number τ_0^k ($k = 1, 2, \dots, m_0$) such that in G_0 the functions $p_0^{k_1}, \dots, p_0^{k_s}$ are a basis for the linear hull of the functions $\{p_0^k\}$ (condition (4) for $i = 0$) and in any region $G^* \subset G_0$ these functions are linearly independent (condition (5*) for $i = 0$). Let $G^* \subset D$ be an arbitrary polyhedral region. Then $\lambda(t, G^*, q, \{p_i^k\})$ as a function of t has, for any $i > 0$, a finite number of points of discontinuity (of the first kind) on the set $q_i(G^*)$, which consists of a finite number of segments (see Lemma 4.3.1). Hence it follows that if the set $\{t : \lambda(t, G^*, q_i, \{p_i^k\}) = 0\}$ is not

nowhere dense on $q_i(G^*)$, then the function $\lambda(t) \equiv 0$ on some segment $\delta \subset q_i(G^*)$ not containing points of Γ_i . By Lemma 4.3.4, there is a segment $\delta^* \subset \delta$ such that in the expression $\sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y))$ one of the terms can be deleted, without narrowing the class of the functions representable in the region $q^{-1}(\delta^*) \cap G^*$ as superpositions of the given form. Carrying out all possible deletions we can find a region $G \subset G_0 \subset D$ for which the assertion of the lemma is satisfied.

A region $G \subset D$ is called regular if, firstly, it is polyhedral and, secondly, there is a number $\gamma_G > 0$ such that for every $i > 0$ and every $t \in q_i(G)$ the set $e(q_i, t) \cap G$ is the union of a finite number of simple arcs, each of which has length not less than γ_G . A point A of the boundary of the polyhedral region G is called a vertex if it belongs simultaneously to two segments of the level curves of $q_i(x, y)$ and $q_j(x, y)$ ($i \neq j$) on the boundary of G . Every polyhedral region has a finite number of vertices.

LEMMA 4.4.2. *For every polyhedral region G and every neighbourhood U of the vertices of this region we can construct a regular region $G^* \subset G$ such that $G \setminus U \subset G^*$.*

Proof. Let A_1, A_2, \dots, A_r be the vertices of the polyhedral region G ; U_1, U_2, \dots, U_r suitably small neighbourhoods of these vertices. Let $k_m = k_m(A_m)$ be the number of all those functions $\{q_i(x, y)\}$ for each of which the level curve passing through the point A_m does not contain any other points of the set $U_m \cap G$. Let $q_{im}(x, y)$ be one of these functions. We put $k(G) \in q_i(G)$. If $k(G) = 0$, then for any i and any $t \in q_i(G)$ the length of any component of the set $e(q_i, t) \cap G$ is greater than zero and consequently the region G is regular. Suppose that $k(G) > 0$ and m such that $k_m \neq 0$.

We fix $\varepsilon > 0$ and put

$$G_{1m}^* = G \setminus \{(x, y): |q_{im}(x, y) - q(A_m)| < \varepsilon\} \cap U_m.$$

If U_m and ε are sufficiently small, then inside U_m the region G_{1m}^* has two vertices A'_m and A''_m , while the region G has only one vertex A_m there, but $k_m(A'_m) = k_m(A''_m) = k_m(A_m) - 1$. We now put $G_1^* = \bigcap G_{1m}^*$, where the intersection is taken over all m such that $k_m \neq 0$. Then $k(G_1^*) = k(G) - 1$. Repeating this construction $k(G)$ times, we obtain a polyhedral region G^* for which $G \setminus G^* \subset U$ and $k(G^*) = 0$. Consequently, G^* is regular. This proves the lemma.

LEMMA 4.4.3. *There exists a set $G \subset D$, a number $\lambda > 0$, and a set of numbers $\tau_i^k = 0$ or 1 ($i=0, 1, \dots, n; k=1, 2, \dots, m_i$) such that condition (4) of Lemma 4.4.1 is satisfied, and also the conditions*

(5) *for every i and $t \in q_i(G)$ and for any functions $\{f_i^k(t)\}$*

$$\max_{(x,y) \in e(q_i,t) \cap G} \left| \sum_{k=1}^{m_i} \tau_i^k p_i^k(x,y) f_i^k(q_i(x,y)) \right| \geq \lambda \max_k |\tau_i^k f_i^k(t)|;$$

(6) *G is a regular region.*

Proof. By Lemma 4.4.1 there exists a region $G^* \subset D$ and a set of numbers τ_i^k such that for every polyhedral subregion $G^{**} \subset G^*$ and for every i the set $\{t: \lambda(t, G^{**}, q_i, p_i^{k_1}, \dots, p_i^{k_s}) = 0\}$ is nowhere dense in $q_i(G^{**})$, where k_1, k_2, \dots, k_s is the set of all values of k for which $\tau_i^k = 1$; moreover, on the set G^* , for any i the property (4) of Lemma 4.4.1 is satisfied. In order not to change the notation unnecessarily, we assume that all $\tau_i^k = 1$. We now construct a system of regular regions $G_0 \supset G_1 \supset G_2 \supset \dots \supset G_n = G$, having the following property: for every $j \leq i$, $\inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_j^k\}) \geq \lambda_i > 0$. For G_0 we choose any regular region $G_0 \in G^*$. Suppose that the regular regions G_0, G_1, \dots, G_{i-1} have been constructed. We now construct the set G_i . We denote by α_δ the set $\{t: \lambda(t, q_i, G_{i-1}, \{p_i^k\}) > \delta\}$. Since the functions $\lambda(t, q_i, G_{i-1}, \{p_i^k\})$, have only finitely many points of discontinuity (of the first kind) on the set $q_i(G_{i-1})$, which consists of a finite number of segments (see Lemma 4.3.1), any component of α_δ is either an interval, or a half-interval, or a segment, or a point. Suppose that the set $\alpha_\delta^N \subset \alpha_\delta$ consists of the N longest components of non-zero length of the set α_δ (if α_δ has only $N_0 (< N)$ components of non-zero length, then let $\alpha_\delta^N = \alpha_\delta^{N_0}$). We denote by $\bar{\alpha}_\delta^N$ the closure of the set α_δ^N . We put $G_{i-1}^* = G_{i-1} \cap q_i^{-1}(\bar{\alpha}_\delta^N)$. We fix $\varepsilon > 0$. Since G_{i-1} is regular, for every j the length of any component of $e(q_j, t) \cap G_{i-1}$ is greater than $\gamma_G > 0$. And since the set $\{t: \lambda(t, q, G_{i-1}, \{p_i^k\}) = 0\}$ is nowhere dense in $q_i(G_{i-1})$, for sufficiently small δ and sufficiently large N the set G_{i-1}^* forms a $\varepsilon/2$ -net on every set $e(q_j, t) \cap G_{i-1}$, $j < i$. The set G_{i-1}^* is a polyhedral region. We denote by $U(\varepsilon)$ the set of points (x, y) each of which is at a distance of no more than $\varepsilon/4$ from one of the vertices of the set G_{i-1}^* . By Lemma 4.4.2 there exists a regular region $G_i \subset G_{i-1}^*$ such that $G_{i-1}^* \setminus G_i \subset U(\varepsilon)$. The set G_i forms an ε -net on every set $e(q_j, t) \cap G_{i-1}$, $j < i$ and forms an $\varepsilon/2$ -net on every set $e(q_i, t) \cap G_{i-1}^*$. By Lemma 4.3.2, for sufficiently small ε ,

$$\lambda_i = \min_{j \leq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2} \min \left\{ \frac{\delta}{2}, \min_{j < i} \lambda_j \right\}.$$

Thus, the regular regions G_1, G_2, \dots, G_n can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma ($\lambda = \lambda_n$), which is now proved.

§ 5. *The set of linear superpositions in the space of continuous functions is closed*

THEOREM 4.5.1. *Suppose that continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m=1, 2, \dots, N$) are fixed. Then in any region D of the plane of the variables x, y , there exists a closed subregion $G \subset D$ such that the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set G .

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ ($i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i; \sum_{i=0}^n m_i \leq N$) that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4') for any continuous functions $\{f_m(t)\}$ there exists continuous functions $\{f_i^k(t)\}$ such that on G

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5') for every i and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x, y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right| \leq \lambda \max_k |f_i^k(t)|;$$

(6') G is a regular region with respect to the functions $\{q_i^k(x, y)\}$.

LEMMA 4.5.1. In the sets $\{q_i^1(G)\}$ we can select subsets consisting of a finite number of points $t_{i,j} \in q_i^1(G)$ ($i=0, 1, 2, \dots, n; j=1, 2, \dots, s_i$) such that for any continuous functions $\{f_i^k(t)\}$

$$\max_{i,k} \max_{t \in q_i^1(G)} |f_i^k(t)| \leq c \left(\max_{(x,y) \in G} \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)) \right| + \max_k |f_i^k(t_{i,j})| \right),$$

where C is a constant not depending on the functions $\{f_i^k(t)\}$.

Proof. Since G is polyhedral, for each i we can choose in $q_i(G)$ a finite set of points $\{t_{i,j}\}$ so dense that the components of the level curves $e(q_i^1, t_{i,j}) \cap G$ form a δ -net in the set of all components of the level curves $e(q_i^1, t) \cap G, t \in q_i^1(G)$. A sufficiently small δ , not depending on the functions $\{f_i^k(t)\}$, will be chosen below. We put

$$\mu = \max_{i,k} \max_{(x,y) \in G} |f_i^k(q_i^1(x,y))|;$$

$$\varepsilon_1 = \max_{(x,y) \in G} \left| \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)) \right|; \quad \varepsilon_2 = \max_{k,i,j} |f_i^k(t_{i,j})|.$$

For definiteness, let $f_1^1(q_1^1(a)) = \mu$ at the point $a \in G$. By (5') there exists a point $a' \in G$ such that $\left| \sum_{k=1}^{m_1} p_1^k(a') f_1^k(q_1^1(a')) \right| \geq \lambda \mu$. Let $[a', a^*]$ be a segment of the level curve of the function $q_1^1(x,y)$ with end-points at a' and a^* such that $h_1([a', a^*]) \geq \gamma G/2$ (see the definition of a regular region in § 4). On the arc $[a', a^*]$ we fix a point a'' such that $\omega(\alpha) \leq \frac{\lambda}{2m_1}$, where

$\alpha = h_1([a', a''])$. Then on the segment $[a', a'']$ the function $\varphi_1(x,y) = \sum_{k=1}^{m_1} p_1^k(x,y) f_1^k(q_1^1(x,y))$ keeps a constant sign and satisfies the inequality $|\varphi_1(x,y)| \geq \lambda \mu/2$. In fact, $|\varphi_1(a')| \geq \lambda \mu$ at the point a' , and for any point $s \in [a', a'']$

$$|\varphi_1(s) - \varphi_1(a')| = \left| \sum_{k=1}^{m_1} (p_1^k(s) - p_1^k(a')) f_1^k(a') \right| \leq m_1 \mu \omega(\alpha) \leq \frac{\lambda \mu}{2}.$$

Consequently,

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds \right| \geq \frac{1}{2} \lambda \mu \alpha.$$

By construction there is an index j and a segment $[b', b'']$ of the level curve $e(q_1^1, t_{1,j}) \cap G$ such that $\rho([a', a''], [b', b'']) < \delta$. We have

$$\left| \int_{s \in [b', b'']} \varphi_1(s) ds \right| \leq c_1 \varepsilon_2 \beta,$$

where $\beta = h_1([b', b''])$, $C_1 = m_1 \max_k \max_{(x,y) \in G} |p_1^k(x,y)|$. And since α and β are commensurable (δ will be chosen small in comparison with α),

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds - \int_{s \in [b', b'']} \varphi_1(s) ds \right| \geq \frac{1}{2} \lambda \mu \alpha - c'_1 \varepsilon_2 \alpha.$$

By Lemma 4.2.3

$$\left| \int_{s \in [a', a'']} \varphi_1(s) ds - \int_{s \in [b', b'']} \varphi_1(s) ds \right| \leq c_3 (\alpha \varepsilon_1 + \mu \alpha \omega(\delta) + \mu \delta).$$

Thus, $c_3 (\alpha \varepsilon_1 + \mu \alpha \omega(\delta) + \mu \delta) \geq \lambda \mu \alpha / 2 - c'_1 \alpha \cdot \varepsilon_2$. If δ is taken sufficiently small in comparison with α (in order that $c_3 (\alpha \omega(\delta) + \delta) < \lambda \alpha / 2$), then we have $\mu \leq C (\varepsilon_1 + \varepsilon_2)$. This proves the lemma.

Let B be the Banach space consisting of all systems of functions $\{f_i^k(t)\}$, defined and continuous on the sets $\{q_i^1(G)\}$, with the norm

$$\|\{f_i^k(t)\}\|_B = \max_{i,k} \max_{t \in q_i^1(G)} |f_i^k(t)| \quad (i=0, 1, 2, \dots, n; k=1, 2, \dots, m_i).$$

We denote by $C(G)$ the space of all functions $f(x,y)$ continuous on G with the uniform metric:

$$\|f(x,y)\|_{C(G)} = \max_{(x,y) \in G} |f(x,y)|.$$

LEMMA 4.5.2. *The linear operator $T: B \rightarrow C(G)$ acting by the formula*

$$T(\{f_i^k(t)\}) = f(x,y) = \sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^1(x,y)),$$

maps bounded closed sets of B onto closed sets of $C(G)$.

Proof. Let $F \subset B$ be a closed and bounded set of elements of B . Suppose that $f_n(x,y)$ is a sequence of functions in $T(F) \subset C(G)$, and that $f(x,y) \in C(G)$, where $\|f(x,y) - f_n(x,y)\|_{C(G)} \rightarrow 0$ as $n \rightarrow \infty$. We show that then $f(x,y) \in T(F)$. Since $f_n(x,y) \in T(F)$, there exists a sequence of elements $\{f_{i,n}^k(t)\} \in F$ such that $T(\{f_{i,n}^k(t)\}) = f_n(x,y)$. By Lemma 4.5.1 we can select in the sets $\{q_i^1(G)\}$ subsets consisting of a finite number of points $t_{i,j} \in q_i^1(G)$ ($i=0, 1, \dots, n; j=1, 2, \dots, s_i$) such that for each element $\{f_i^k(t)\} \in B$ the inequality

$$\|\{f_i^k(t)\}\|_B \leq c (\|f(x,y)\|_{C(G)} + \max_{k,j,i} |f_i^k(t_{i,j})|),$$

is satisfied, where the constant C does not depend on the functions $\{f_i^k(t)\}$. Since F is a bounded set, there exists a subsequence of suffixes n_1, n_2, \dots such that for any $i = 0, 1, \dots, n$; $k = 1, 2, \dots, m_i$; $j = 1, 2, \dots, s_i$ the numerical sequence $f_{i,n_\nu}^k \rightarrow C_{k,i,j}$ as $\nu \rightarrow \infty$. From this and the previous inequality it follows that $\{f_{i,n_\nu}^k(t)\} \in F (\nu = 1, 2, \dots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f_i^k(t)\} \in B$ such that $\|\{f_i^k(t) - f_{i,n_\nu}^k(t)\}\|_B \rightarrow 0$. Since F is a closed set, $\{f_i^k(t)\} \in F$. The operator $T: B \rightarrow C(G)$ is bounded. Therefore $T(\{f_i^k(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let B_1 and B_2 be Banach spaces. If a linear operator $T: B_1 \rightarrow B_2$ maps bounded closed sets of B_1 onto closed sets of B_2 , then its domain of values is closed.*

Proof of Theorem 4.5.1. The set of superpositions of the form $\sum_{m=1}^N p_m(x, y) f_m(g_m(x, y))$ coincides on G with the set of superpositions of the form $\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

§ 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ ($m = 1, 2, \dots, N$) and any region D of the plane of the variables x, y the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$

where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in D with uniform convergence.

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that

the functions $\tilde{q}_i^k(x, y)$ ($i=0, 1, 2, \dots, \tilde{n}; k=1, 2, \dots, \tilde{m}_i; \sum_{i=0}^{\tilde{n}} \tilde{m}_i = N$) obtained after the renumbering satisfy conditions (1), (2), (3) of Lemma 4.2.2. We now fix the point $(x_0, y_0) \in G^*$ and the number v so that the line $(y - y_0) + v(x - x_0) = 0$ does not touch at any of the level curves of the functions $\tilde{q}_i^k(x, y)$ ($i=1, 2, \dots, \tilde{n}$) that pass through (x_0, y_0) . Let $G^{**} \subset G^*$ be a disc with centre at (x_0, y_0) and radius small enough so that the $\{\tilde{q}_i^k(x, y)\}$ and $q_{N+1}(x, y) = y + vx$ satisfy condition (3) of Lemma 4.2.2 with some constant $\gamma^{**} > 0$. We put $p_{N+1}(x, y) = 1$. By Lemma 4.4.3 we can find a set $G \subset G^{**}$, determine a constant $\lambda > 0$, and again renumber the functions $p_m(x, y)$ and $q_m(x, y)$ ($m=1, 2, \dots, N+1$) with two indices so that the functions $p_i^k(x, y)$ and

$$q_i^k(x, y) \quad (i=0, 1, 2, \dots, n+1; k=1, 2, \dots, m_i; \sum_{i=0}^{n+1} m_i \leq N+1)$$

that is, some functions may be omitted in the renumbering) obtained after the renumbering satisfy conditions (1)-(3) of Lemma 4.2.2, conditions (4')-(6') of § 5, and the condition

$$7 \quad m_{n+1} = 1, \quad p_{N+1}^1 = p_{N+1}(x, y) = 1, \quad q_{N+1}^1 = q_{N+1}(x, y) = y + vx.$$

Let L be the linear space consisting of all system of functions $\{f_i^k(t)\}$ defined and continuous on the sets $\{q_i^1(G)\}$ and satisfying the condition

$$\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \equiv 0 \quad \text{in } G.$$

LEMMA 4.6.1. L is a finite-dimensional linear space.

Proof. By Lemma 4.5.1, in the sets $\{q_i^1(G)\}$ we can select a subset consisting of a finite number of points $\{t_{i,j}\}$ such that, if $\{f_i^k(t)\} \in L$ and $f_i^k(t_{i,j}) = 0$ for all k, i, j then $f_i^k(t) \equiv 0$ on $q_i^1(G)$ for all i, k . Thus, the set of functions $\{f_i^k(t)\}$ is completely determined by a finite set of parameters $\{f_i^k(t_{i,j})\}$. Consequently the dimension of the space L is finite. This proves the lemma.

LEMMA 4.6.2. There exists a natural number μ such that in D the polynomial $(y + vx)^\mu = Q(x, y)$ is not equal to any superposition of the form

$$\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y)),$$
 where $\{f_m(t)\}$ are arbitrary continuous functions.

Proof. We denote by Φ the space of functions of the form $f(y + vx) = f_{n+1}^1(q_{n+1}^1(x, y))$ that are representable on G by superpositions of the form $[\sum_{m=1}^N p_m(x, y) f_m(q_m(x, y))]$. Or, what comes to the same thing

(see properties (4') and (7)), of the form $[\sum_{i=0}^n \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y))]$.

Thus, functions of Φ satisfy the relation $\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \equiv 0$

in G . Consequently the linear space Φ is naturally embedded in L . Since L is finite-dimensional (Lemma 4.6.1), Φ is also finite-dimensional. Let l be the dimension of Φ . Since the polynomials $(y + vx), (y + vx)^2, \dots, (y + vx)^{l+1}$ are linearly independent, at least one of them $Q(x, y) = (y + vx)^\mu$ is not equal to any superposition of the form under discussion on G or, consequently, in D . This proves the lemma.

Proof of Theorem 4.6.1. By Lemma 4.6.2 the set of superpositions of the form given in Theorem 4.6.1 does not exhaust all continuous functions on G . Consequently, by Theorem 4.5.1, the set of these superpositions is a closed linear subspace of $C(G)$. Hence we conclude that the set of superpositions under discussion is nowhere dense in $C(G)$, nor consequently in $C(D)$. This proves the theorem.

COROLLARY 4.6.1. *For any continuous functions $p_m(x_1, x_2, \dots, x_n)$ and continuously differentiable functions $q_m(x_1, x_2, \dots, x_n)$ ($m=1, 2, \dots, N$) and any region D of the space of the variables (x_1, x_2, \dots, x_n) the set of superpositions of the form*

$$\sum_{m=1}^N p_m(x_1, x_2, \dots, x_n) f_m(q_m(x_1, x_2, \dots, x_n), x_2, x_3, \dots, x_{n-1}),$$

where $\{f_m(t, x_2, x_3, \dots, x_{n-1})\}$ are arbitrary continuous functions of $(n-1)$ variables, is nowhere dense in the space of all functions continuous in D with uniform convergence.