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## CHAPTER 4. — LINEAR SUPERPOSITIONS

In this chapter we prove that there exist analytic functions which are not representable by means of linear superpositions of smooth functions of one variable.

### § 1. *Notation*

Throughout we assume that all the functions are defined and continuous for all values of the arguments. If we say that a function is continuously differentiable, we mean by this that its first partial derivatives are defined and continuous for all values of the arguments; $z = (x, y)$ is the point of the plane with coordinates $x$ and $y$; grad $[q(z)]$ is the gradient of the function $q(z)$, that is, the vector-function with coordinates $\frac{\partial q}{\partial x}$ and $\frac{\partial q}{\partial y}$;

$$D\left(\frac{q_1, q_2}{x, y}\right) = \begin{vmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} \end{vmatrix}$$

is the Jacobian of the pair of functions $q_1$ and $q_2$.

$q(D)$ is the image of the set $D$ under the mapping effected by the function $q(x, y)$; $q^{-1}(\delta)$ is the complete inverse image of the interval $\delta$ on the axis of values of the function $q(x, y)$.

$e(q, t)$ is the set of level $t$ of the function $q = q(x, y)$.

$\tau(e, z)$ is the unit tangent vector to the curve $e$ at the point $z \in e$.

$\gamma(\tau_1, \tau_2)$ is the absolute value of the acute angle between the vectors $\tau_1$ and $\tau_2$.

$h_1(e)$ is the length of the set $e$.

$d_1(e)$ is the one-dimensional diameter of the set $e$.

$0(\gamma)$ is a quantity bounded by a constant depending only on $\gamma$.

$\rho(A_1, A_2)$ is the distance between the sets $A_1$ and $A_2$ in the sense of deviation, more precisely

$$\rho(A_1, A_2) = \max\left\{\sup_{z_1 \in A_1} \inf_{z_2 \in A_2} \rho(z_1, z_2), \sup_{z_2 \in A_2} \inf_{z_1 \in A_1} \rho(z_1, z_2)\right\},$$

where $\rho(z_1, z_2)$ is the distance between the points $z_1$ and $z_2$.

## § 2. *Estimate of the difference of the integrals of one term of a superposition along nearby level curves*

Let $G$ be a region of the plane of the variables $x$ and $y$, and $q_1(x, y)$ and $q_2(x, y)$ continuously differentiable functions satisfying in this region the following conditions: a) the partial derivatives with respect to $x$ and with respect to $y$ have modulus of continuity $\omega(\delta)$; b) the inequalities

$$0 < \gamma \leqslant |\text{ grad } [q_i(x, y)]| \leqslant \frac{1}{\gamma} < \infty \qquad (i = 1, 2)$$

are satisfied everywhere in $G$, where $\gamma$ is a constant; c) for any point $(x, y) \in G$ the absolute value of the acute angle formed by the level curves of the functions $q_1(x, y)$ and $q_2(x, y)$ which pass through this point is greater than some positive constant $\gamma$.

LEMMA 4.2.1. *Let $e'_{q2}$ and $e''_{q2}$ be two level curves of the function $q_2$ and $e'_{q1}$ and $e''_{q1}$ level curves of the function $q_1$; $[a', a''] \subset G$ the segment of the curve $e'_{q1}$ with end-points $a' \in e'_{q2}$ and $a'' \in e''_{q2}$; $[b', b'']$ the segment of the curve $e''_{q1}$ with end-points $b' \in e'_{q2}$ and $b'' \in e''_{q2}$. Then*

$$h_1([b', b'']) \leqslant h_1([a', a'']) \times (1 + c_1(\gamma)\,\omega(\delta)),$$

*where $\delta = d_1([a', a''] \cup [b', b''])$ and $c_1(\gamma)$ depends only on $\gamma$.*

*Proof.* Since $q_2(a'') - q_2(a') = q_2(b'') - q_2(b')$, we have

$$\int\limits_{s \in [a', a'']} \frac{\partial q_2}{\partial s}\, ds = \int\limits_{s \in [b', b'']} \frac{\partial q_2}{\partial s}\, ds .$$

Consequently, $\dfrac{\partial q_2(a^*)}{\partial s}\, h_1([a', a'']) = \dfrac{\partial q_2(b^*)}{\partial s}\, h_1([b', b''])$, where $\dfrac{\partial q_2(a^*)}{\partial s}$ and $\dfrac{\partial q_2(b^*)}{\partial s}$ are the derivatives at the points $a^* \in [a', a'']$ and $b^* \in [b', b'']$ along the curves $[a', a'']$ and $[b', b'']$, respectively. We show that $\dfrac{\partial q_2(a^*)}{\partial s} = \dfrac{\partial q_2(b^*)}{\partial s} + O(\gamma)\,\omega(\delta)$. We denote by $q_2^*$ the derivative of $q_2$ at the point $b^*$ in the direction of $\tau(e'_{q1}, a^*)$ and put $\alpha = \gamma\{\tau[e''_{q1}, b^*], \tau[e'_{q1}, a^*]\}$. From conditions a) and b) it follows that $\dfrac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1)\,\omega(\delta)$ and $\alpha$

$= O(\gamma)\,\omega(\delta)$. We denote by $\beta_1$ and $\beta_2$ the values of the angles formed by the vectors $\tau\,[e''_{q_1}, b^*]$ and $\tau\,[e'_{q_1}, a^*]$ with the vector grad $[q_2(b^*)]$. We have

$$\left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| = |\operatorname{grad}[q_2(b^*)]|\;|\cos\beta_2 - \cos\beta_1| = O(\gamma)\,\alpha$$
$$= O(\gamma)\,\omega(\delta)\,.$$

Thus,

$$\frac{\partial q_2(a^*)}{\partial s} = q_2^* + O(1)\,\omega(\delta) = \frac{\partial q_2(b^*)}{\partial s}$$
$$+ O(1)\left\{ \left| q_2^* - \frac{\partial q_2(b^*)}{\partial s} \right| + \omega(\delta) \right\} = \frac{\partial q_2(b^*)}{\partial s} + O(\gamma)\,\omega(\delta)\,.$$

Consequently,

$$h_1([b', b'']) = h_1([a', a''])\,\frac{\partial q_2(a^*)}{\partial s}\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1}$$
$$= h_1([a', a''])\left(1 + O(\gamma)\,\omega(\delta)\left(\frac{\partial q_2(b^*)}{\partial s}\right)^{-1}\right)$$
$$= h_1([a', a''])\,(1 + O(\gamma)\,\omega(\gamma))\,,$$

since by virtue of b) $\dfrac{\partial q_2(b^*)}{\partial s} > |\operatorname{grad}[q_2(b^*)]|\sin\gamma$. This, proves the lemma.

LEMMA 4.2.2. *Let* $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ *be continuously differentiable functions. In any region* $D$ *we can find a subregion* $G \subset D$, *determine a constant* $\gamma > 0$, *and renumber the functions* $\{ q_m(x, y) \}$ *with two indices so that the functions*

$$q_i^k(x, y) = q_m(x, y) \quad (i = 0, 1, 2, \ldots, n\,;\; k = 1, 2, \ldots, m_i\,;\; \sum_{i=0}^{n} m_i = N)$$

*obtained after the renumbering satisfy the following conditions:*

(1) *when* $i = 0$, $q_i^k = \text{const}$ *in* $G$, *and when* $i > 0$, $\gamma \leqslant |\operatorname{grad}[q_i^k(x, y)]| \leqslant \dfrac{1}{\gamma}$ *for every point* $(x, y) \in G$;

(2) *the functions* $q_i^k(x, y)$ $(i > 0$ *fixed,* $k = 1, 2, \ldots, m_i)$ *have in the region* $G$ *identical sets of level curves, more precisely, in the region* $G$, $q_i^k(x, y) \equiv \varphi_i^{k,l}(q_i^l(x, y))$, *where* $\varphi_i^{k,l}(t)$ *is a strictly monotonic continuously differentiable function of* $t$;

(3) *when* $i \neq j\,(i, j \neq 0)$, *then for any* $k$ *and* $l$ *the absolute value of the acute angle formed by the level curves of the functions* $q_i^k(x, y)$ *and* $q_j^l(x, y)$ *which pass through an arbitrary point* $(x, y) \in G$ *is greater than* $\gamma$.

*Proof.* By the continuity of the partial derivatives of the functions $\{ q_m(x, y) \}$ there exists a subregion $G^* \subset D$ inside which for any function $q_m(x, y)$ either grad $q_m(x, y) \equiv 0$ or $|$ grad $q_m(x, y) |$ is greater than some positive constant. From the continuity of the partial derivatives of the functions $\{ q_m(x, y) \}$ it follows also that there exists a subregion $G^{**} \subset G^*$ inside which for any pair of functions $q_r(x, y)$ and $q_s(x, y)$ one of two conditions holds: either $D\left(\dfrac{q_r, q_s}{x, y}\right) \equiv 0$ in $G^{**}$, or for every point of $G^{**}$ the level curves of $q_r(x, y)$ and $q_s(x, y)$ that pass through this point intersect at a non-zero angle $\left(D\left(\dfrac{q_r, q_s}{x, y}\right) \neq 0 \text{ in } G^{**}\right)$. From the implicit function theorem it follows that there exists a subregion $G \subset G^{**}$ in which condition (2) is satisfied for every pair of functions $q_r(x, y)$ and $q_s(x, y)$ with gradients different from zero and with determinant $D\left(\dfrac{q_r, q_s}{x, y}\right) \equiv 0$.

We now renumber the functions $\{ q_m(x, y) \}$ with two indices in such a way that only functions constant in $G$ have lower index zero, and the same lower index is assigned to those functions whose level curves coincide identically in $G$. This proves the lemma.

We consider in the region $G$ a superposition of the form $\sum\limits_{i=0}^{n} \sum\limits_{k=1}^{m_i} p_r(x, y)$ $f_i^k(q_i^k(x, y))$, where $\{ f_i^k(t) \}$ are continuous functions of one variable, $\{ p_i^k(x, y) \}$ are continuous functions satisfying in $G$ the condition $| p_i^k(x, y) | \leqslant \dfrac{1}{\gamma}$ and $\{ q_i^k(x, y) \}$ are continuously differentiable functions satisfying in $G$ conditions (1), (2), (3) of Lemma 4.2.2. Let $\omega(\delta)$ be the common modulus of continuity in $G$ of the functions $\left\{ p_i^k(x, y) ; \dfrac{\partial q_i^k(x, y)}{\partial x} ; \dfrac{\partial q_i^k(x, y)}{\partial y} \right\}$. Let $[a', a'']$ and $[b', b'']$ be segments of the level curves of the functions $\{ q_i^k(x, y) \}$ $(i > 0$ fixed) lying in $G$. Let

$$\alpha = h_1([a', a'']) ; \ \delta = \rho([a', a''], \ [b', b'']) ;$$

$$\varepsilon = \sup \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y)) \right| ;$$

$$m = \max_{i,k} \sup \left| f_i^k (q_i^k (x, y)) \right| ,$$

where sup is taken over all points $(x, y) \in [a', a''] \cup [b', b'']$.

LEMMA 4.2.3. *If* $\delta$ *is sufficiently small* $(\omega(\delta) \leqslant C_2(\gamma))$, *then for any* $i > 0$

$$\left| \int_{s \in [a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k (q_i^k(s))\, ds - \int_{s \in [b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k (q_i^k(s))\, ds \right|$$
$$\leqslant C_3(\gamma)(\alpha\varepsilon + m\alpha\omega(\delta) + m\delta),$$

*where the constants* $C_2(\gamma)$, $C_3(\gamma)$ *depend only on* $\gamma$.

*Proof.* By (1), (2), (3) there exists a sufficiently small constant $C_2(\gamma)$ and a sufficiently large constant $C_3(\gamma)$ such that if $\omega(\delta) \leqslant C_2(\gamma)$ and for a point $a \in [a', a'']$ the inequalities $h_1([a', a]) \geqslant C_3(\gamma)\,\delta$; $h_1([a, a'']) \geqslant C_3(\gamma)\delta$ are satisfied, then for any $j \neq i$ $(j > 0)$ the level curve of the function $q_j^k$ that passes through $a$ intersects $[b', b'']$ of the level curve of $q_i^k$. Suppose that $\alpha > 2C_3(\gamma)\,\delta$ (if $\alpha \leqslant 2\,C_3(\gamma)\,\delta$, then the assertion of the lemma is trivial) and suppose that the segment $[\tilde{a}', \tilde{a}'']$ of the level curve of $q_i^k$ is such that $[\tilde{a}', \tilde{a}''] \subset [a', a'']$ and $h_1([a', \tilde{a}']) = h_1([\tilde{a}'', a'']) = C_3(\gamma)\,\delta$. On the arc $[\tilde{a}', \tilde{a}'']$ we fix a system of points $a_1, a_2, \ldots, a_\nu$ $(\tilde{a}' = a_1,\ \tilde{a}'' = a_\nu)$, uniformly distributed along the length of this arc, and denote by $b_r$ the point of intersection of $[b', b'']$ with the level curve of $q_j^k$ that passes through $a_r$ (here $j \neq i$ should for the time being be regarded as fixed). Using Lemma 4.2.1 we have

$$\left| \int_{s \in [a', a'']} p_j^k(s) f_j^k (q_j^k(s))\, ds - \int_{s \in [b', b'']} p_j^k(s) f_j^k (q_j^k(s))\, ds \right|$$

$$= \left| \int_{s \in [a_1, a_\nu]} p_j^k(s) f_j^k (q_j^k(s))\, ds - \int_{s \in [b_1, b_\nu]} p_j^k(s) f_j^k (q_j^k(s))\, ds \right|$$
$$+ O(\gamma)\, m\delta$$

$$= \lim_{\nu \to \infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k (q_j^k(a_r))\, h_1([a_r, a_{r+1}])$$
$$- \sum_{r=1}^{\nu} p_j^k(b_r) f_j^k (q_j^k(b_r))\, h_1([b_r, b_{r+1}]) \Big| + O(\gamma)\, m\delta$$

$$= \lim_{\nu\to\infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])$$

$$- \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])\,(1 + O(\gamma)\,\omega(\delta))$$

$$+ \sum_{r=1}^{\nu} (p_j^k(a_r) - p_j^k(b_r)) f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}]) \Big| + O(\gamma)\, m\delta$$

$$= \lim_{\nu\to\infty} \Big| \sum_{r=1}^{\nu} p_j^k(a_r) f_j^k(q_j^k(a_r))\, h_1([a_r, a_{r+1}])\, O(\gamma)\,\omega(\delta)$$

$$+ \sum_{r=1}^{\nu} f_j^k(q_j^k(a_r))\, h_1([b_r, b_{r+1}])\, O(\gamma)\,\omega(\delta) \Big| + O(\gamma)\, m\delta$$

$$= O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\alpha\omega(\delta) + O(\gamma)\, m\delta = O(\gamma)\, m(\delta + \alpha\omega(\delta))\,.$$

Then

$$\Big| \int\limits_{s\in[a', a'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds$$

$$\leqslant \Big| \int\limits_{s\in[a', a'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(s) f_i^k(q_i^k(s))\, ds \Big|$$

$$+ \sum_{j\neq i} \Big| \int\limits_{s\in[a', a'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds - \int\limits_{s\in[b', b'']} \sum_{k=1}^{m_j} p_j^k(s) f_j^k(q_j^k(s))\, ds \Big|$$

$$\leqslant C_4(\gamma)\,\alpha\varepsilon + n\,(\max_{j\neq i}\ m_j)\, C_5(\gamma)\, m\,(\delta + \alpha\omega(\delta))$$

$$\leqslant C_3(\gamma)\,\big(\alpha\varepsilon + m\delta + m\alpha\omega(\delta)\big)\,.$$

This proves the lemma.

## § 3. *Deletion of dependent terms*

On a bounded closed set $D$ we consider the space of linear superpositions of the form $\sum_{k=1}^{m} p_k(x, y) f_k(q(x, y))$, $(x, y) \in D$. Here the functions $\{p_k(x, y)\}$ and $q(x, y)$ are continuous and fixed, and $\{f_k(t)\}$ are arbitrary continuous functions of one variable. We assume that the function $q(x, y)$ is such that for any sequence $t_n \in q(D) \to t \in q(D)$ we have $\rho\,[e(q, t_n) \cap D, e(q, t) \cap D] \to 0$. We put

$$\lambda(t, D, q, p_1, \dots, p_m) = \inf_{\{c_k\}} \sup_{(x,y)\in e(q,t)\cap D} \Big| \sum_{k=1}^{m} c_k p_k(x, y) \Big|\,,$$

where inf is taken over all sets of numbers $\{ c_k \}$ for which $\max_k | c_k | = 1$. The function $\lambda (t, D, q, \{ p_k \})$, as a function of $t$, is defined only on the set $q (D)$.

LEMMA 4.3.1. *The function* $\lambda (t, D, q, \{ p_k \})$ *depends continuously on* $t$.

*Proof.* The linear combinations $\sum_{k=1}^{m} c_k p_k (x, y)$ for all possible systems of numbers $\{ c_k \}$ for which $\max_k | c_k | \leqslant 1$, form an equicontinuous set of functions, considered on the bounded closed set $D$. Consequently, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $| t_1 - t_2 | < \delta$, then

$$\Big| \sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k p_k (x, y) \Big| - \sup_{(x, y) \in e(q, t_2)} \Big| \sum_{k=1}^{m} c_k p_k (x, y) \Big\| < \varepsilon$$

simultaneously for all systems of numbers $\{ c_k \}$ such that $\max_k | c_k | \leqslant 1$. For definiteness, suppose that $\lambda (t_2, D, q, \{ p_k \}) \geqslant \lambda (t_1, D, q, \{ p_k \})$. Since the expression $\sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k p_k (x, y) \Big|$ depends continuously on the coefficients $\{ c_k \}$, there exists a system of numbers $\{ c_k^1 \}$ such that $\max_k | c_k^1 | = 1$ and

$$\lambda (t_1, D, q, \{ p_k \}) = \sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big| .$$

Since

$$\lambda (t_2, D, q, \{ p_k \}) \leqslant \sup_{(x, y) \in e(q, t_2)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big| ,$$

we have

$$0 \leqslant \lambda (t_2) - \lambda (t_1) \leqslant \sup_{(x, y) \in e(q, t_2)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big|$$

$$- \sup_{(x, y) \in e(q, t_1)} \Big| \sum_{k=1}^{m} c_k^1 p_k (x, y) \Big| < \varepsilon .$$

This proves the lemma.

LEMMA 4.3.2. *The function* $\lambda (t, D, q, \{ p_k \})$ *depends continuously on* $D$ *in the sense that there exists a function* $\mu (\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *having the property: if the set* $D_\varepsilon \subset D$ *is such that, for any* $t$, $D_\varepsilon \cap e (q, t)$ *forms an* $\varepsilon$*-net in the set* $e (q, t) \cap D$, *then*

$$\max_{t \in q(D)} \left| \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \right| \leqslant \mu(\varepsilon) .$$

*Proof.* Using the equicontinuity of the set of functions $\sum_{k=1}^{n} c_k\, p_k(x, y)$ where $\max_k |c_k| \leqslant 1$, we conclude that there exists a function $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that the inequality

$$0 \leqslant \sup_{(x, y) \in e(q,t) \cap D} \left| \sum_{k=1}^{m} c_k p_k(x, y) \right| - \sup_{(x,y) \in e(q,t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k p_k(x, y) \right| \leqslant \mu(\varepsilon).$$

uniformly over all $t \in q(D)$ and over all systems of numbers $\{c_k\}$ for which $\max_k |c_k| \leqslant 1$. For any $\varepsilon > 0$ there exists a system of numbers $\{c_k^\varepsilon\}$ such that $\max_k |c_k^\varepsilon| = 1$ and

$$\lambda(t, D_\varepsilon, q, \{p_k\}) = \sup_{(x,y) \in e(q,t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right| .$$

Since for any $\varepsilon$

$$\lambda(t, D, q, \{p_k\}) \leqslant \sup_{(x,y) \in e(q,t) \cap D} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right|$$

and, on the other hand, $\lambda(t, D, q, \{p_k\}) \geqslant \lambda(t, D_\varepsilon, q, \{p_k\})$ (we recall that $D_\varepsilon \subset D$), we have

$$0 \leqslant \lambda(t, D, q, \{p_k\}) - \lambda(t, D_\varepsilon, q, \{p_k\}) \leqslant \sup_{(x,y) \in e(q,t) \cap D} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right|$$
$$- \sup_{(x,y) \in e(q,t) \cap D_\varepsilon} \left| \sum_{k=1}^{m} c_k^\varepsilon p_k(x, y) \right| < \mu(\varepsilon) .$$

This proves the lemma.

LEMMA 4.3.3. *Let $F$ be a closed set on the $t$-axis; $F \subset q(D)$. For every $t \in F$, suppose that there exists one and only one system of numbers $\{C_k\}$ $(\max_k |C_k| = 1)$ such that $\sum_{k=1}^{m} C_k p_k(x, y) \equiv 0$ on the set $e(q, t) \cap D$. Then each of the functions $\{C_k(t)\}$ depends continuously on $t$ on the set $F$.*

*Proof.* Suppose that $t_n \in F$, $t \in F$ and $t_n \to t$. We put $\overline{\lim_{n \to \infty}} C_k(t_n) = \tilde{C}_k$ and $\underline{\lim_{n \to \infty}} C_k(t_n) = \tilde{\tilde{C}}_k$. Since $\sum_{k=1}^{m} C_k(t_n)\, p_k(x, y) \equiv 0$ on the set $e(q, t_n) \cap D$ and $\rho\,[e(q, t) \cap D, e(q, t_n) \cap D] \to 0$ as $n \to \infty$, we have $\sum_{k=1}^{m} \tilde{C}_k\, p_k(x, y)$

$\equiv 0 \equiv \sum_{k=1}^{m} \overset{\approx}{C}_k p_k(x, y)$ on the set $e(q, t) \cap D$. Consequently, by the condition of the lemma, $\tilde{C}_k = \overset{\approx}{C}_k = C_k(t)$. This proves the lemma.

LEMMA 4.3.4. *Suppose that* $\lambda(t, D, q, \{p_k\}) \equiv 0$ *on some non-empty portion* $\delta$ *of the set* $q(D)$. *Then there is a non-empty portion* $\delta^* \subset \delta$ *and an index* $l$ *such that for any continuous functions* $\{f_k(t)\}$ *there are continuous functions* $\{f_k^*(t)\}$ *such that*

$$\sum_{k \neq l} f_k^*(q(x, y)) p_k(x, y) = \sum_{k=1}^{m} f_k(q(x, y)) p_k(x, y)$$

*on the set* $q^{-1}(\delta^*) \cap D$.

We recall that a portion $\delta$ of a set $E$ is that part of it which lies in the interval $\delta$.

*Proof.* We prove the lemma by induction on $m$. For $m = 1$ the assertion of the lemma is obvious. We denote by $\delta_k$ the set of all points $t$ of the portion $\delta$ for which $\lambda(t, D, q, p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_m) = 0$. By Lemma 4.3.1, the set is closed. Two cases are possible.

1) For some $k$ the set $\delta_k$ contains a non-empty portion $\delta_k'$ of the set $q(D)$. Since $\lambda(t, D, q, p_1, \ldots, p_{k-1}, p_{k+1}, \ldots, p_m) = 0$ for every $t \in \delta_k'$, then by the inductive hypothesis there is a non-empty portion $\delta^* \subset \delta_k'$ and an index $l \neq k$ such that for any continuous functions $f_1(t), \ldots, f_{k-1}(t), f_{k+1}(t), \ldots, f_m(t)$ there are continuous functions $f_1^*(t), \ldots, f_{k-1}^*(t), f_{k+1}^*(t), \ldots, f_m^*(t)$ such that

$$\sum_{i \neq k} f_i(q(x, y)) p_i(x, y) = \sum_{i \neq k, l} f_i^*(q(x, y)) p_i(x, y).$$

on the set $q^{-1}(\delta^*) \cap D$. Putting $f_k^*(t) = f_k(t)$, we obtain

$$\sum_{i=1}^{m} f_i(q(x, y)) p_i(x, y) = \sum_{i \neq l} f_i^*(q(x, y)) p_i(x, y).$$

So in case 1) the lemma is proved.

2) None of the sets $\delta_k$ contains non-empty portions of the set $q(D)$, that is, $\bigcup_{k=1}^{m} \delta_k$ is nowhere dense in $q(D)$. Therefore there exists a non-empty portion $\delta^* \subset \delta \setminus \bigcup_{k=1}^{m} \delta_k$. Since $\lambda(t, D, q, \{p_k\}) \equiv 0$ on $\delta^*$, for every $t \in \delta^*$ there are numbers $\{C_k(t)\}$ ($\max_k |C_k(t)| = 1$) such that $\sum_{k=1}^{m} C_k$

$(q(x, y))\, p_k(x, y) \equiv 0$ on $e(q, t) \cap D$. If we had $C_k(t) = 0$ for some $k$, then it would turn out that $t \in \delta_k$. Consequently, $C_k(t) \neq 0$ for any $k$. We show that for every $t \in \delta^*$ the numbers $\{ C_k(t) \}$ are uniquely determined. Assume the contrary. Then there are numbers $\{ C_k'(t) \}$ $(\max | C_k'(t) | = 1)$ such that $\sum_{k=1}^{m} C_k'(q(x, y))\, p_k(x, y) = 0$ on $e(q, t) \cap D$ and $C_k \neq C_k'$ for some $k$. Then

$$\sum_{k \neq 1} \left[C_k(t)\, C_1'(t) - C_k'(t)\, C_1(t)\right] p_k(x, y) = \sum_{k \neq 1} C_k''(t)\, p_k(x, y) \equiv 0$$

on $e(q, t) \cap D$ and in addition, $C_k'' \neq 0$ for some $k$. Consequently, $t \in \delta_1$. So we have obtained a contradiction, and the uniqueness of the choice of the numbers $C_k(t)$ is proved. Further, we may regard $\{ C_k(t) \}$ as single-valued functions of $t$ on the portion $\delta^*$. By Lemma 4.3.3, the functions $C_k(t)$ are continuous and, as noted above, $C_k(t) \neq 0$ for any $t \in \delta^*$. Then

$$p_1(x, y) = \sum_{k=2}^{m} - \frac{C_k(q(x, y))}{C_1(q(x, y))}\, p_k(x, y)\,, \quad (x, y) \in q^{-1}(\delta^*) \cap D\,.$$

Putting $f(t) = f_k(t) - \dfrac{C_k(t)}{C_1(t)} f_1(t)$, $t \in \delta^*$, we have $\sum_{k=2}^{m} f_k^*(q(x, y))\, p_k(x, y)$

$$= \sum_{k=1}^{m} f_k(q)\, p_k(x, y) - \sum_{k=2}^{m} \frac{C_k(q)}{C_1(q)}\, p_k(x, y)$$

$$= \sum_{k=2}^{m} f_k(q)\, p_k(x, y) + f_1(q)\, p_1(x, y)$$

$$= \sum_{k=1}^{m} f_k(q(x, y))\, p_k(x, y)\,, \quad (x, y) \in q^{-1}(\delta^*) \cap D\,.$$

This proves the lemma.

## § 4. *Reduction of linear superpositions to a form with independent terms*

We fix the continuous functions $p_i^k(x, y)$ and continuously differentiable functions $q_i(x, y)$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i)$ $n \geqslant 2$, where $\{ q_i(x, y) \}$ satisfy in $D$ conditions (1) and (3) of Lemma 4.2.2, and we consider in $D$ superpositions of the form

$$\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y))\,,$$

where $\{ f_i^k(t) \}$ are arbitrary continuous functions of one variable.

We call a bounded closed region $G \subset D$ polyhedral if the boundary of $G$ consists of a finite number of mutually non-intersecting simple closed contours that are unions of a finite number of segments of level curves of the functions $q_i(x, y)$ $(i = 1, 2, \ldots, n)$. Let $G \subset D$ be a polyhedral region. We denote by $\Gamma_i$ the set of those $t \in q_i(G)$ for which the set $e(q_i, t) \cap G$ contains a segment of a level curve belonging to the boundary of $G$. For any $i$ the set $\Gamma_i$ consists of a finite number of points. By property (1) of the functions $\{ q_i(x, y) \}$ for every $i$ and for all points $t_0 \in q_i(G) \backslash \Gamma_i$ there exists $\lim\limits_{t \to t_0} e(q_i, t) \doteqdot e(q_0, t_0)$. If $t_0 \in \Gamma_i$, then the last assertion need not hold, but in any case there exists $\lim\limits_{t \to +t_0} e(q_i, t) \subset e(q_i, t_0)$ and $\lim\limits_{t \to -t_0} e(q_i, t) \subset e(q_i, t_0)$ where the limit is taken over the points $t \in q_i(G)$. Here the limit is understood in the sense of the distance $\rho\big(e(q_i, t), e(q_i, t_0)\big)$.

Lemma 4.4.1. *There is a region* $G \subset D$ *and a system of numbers* $\tau_i^k = 0$ *or* $1$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i)$ *such that*

(4) *for any* $i$ *and for any continuous functions* $\{ \varphi_i^k(t) \}$ *there exist continuous functions* $\{ f_i^k(t) \}$ *such that in* $G$

$$\sum_{k=1}^{m_i} p_i^k(x, y)\, \varphi_i^k(q_i(x, y)) \equiv \sum_{k=1}^{m_i} \tau_i^k p_i^k(x, y) f_i^k(q_i(x, y))\,;$$

(5*) *for any polyhedral region* $G^* \subset G$ *and any* $i$, *the set*

$$\{ t : \lambda(t, G^*, q_i, p_i^{k_1}, \ldots, p_i^{k_s}) = 0 \}$$

*is nowhere dense in* $q_i(G^*)$, *where*

$$k_1 = k_1(i), k_2 = k_2(i), \ldots, k_s = k_s(i)$$

*is the set of all values of* $k$ *for which* $\tau_i^k = 1$.

*Proof.* If $i = 0$, then by (1) the set $q_0(D)$ consists of only one point. We choose a region $G_0 \subset D$ and number $\tau_0^k$ $(k = 1, 2, \ldots, m_0)$ such that in $G_0$ the functions $p_0^{k_1}, \ldots, p_0^{k_s}$ are a basis for the linear hull of the functions $\{ p_0^k \}$ (condition (4) for $i = 0$) and in any region $G^* \subset G_0$ these functions are linearly independent (condition (5*) for $i = 0$). Let $G^* \subset D$ be an arbitrary polyhedral region. Then $\lambda(t, G^*, q, \{ p_i^k \})$ as a function of $t$ has, for any $i > 0$, a finite number of points of discontinuity (of the first kind) on the set $q_i(G^*)$, which consists of a finite number of segments (see Lemma 4.3.1). Hence it follows that if the set $\{ t : \lambda(t, G^*, q_i, \{ p_i^k \}) = 0 \}$ is not nowhere dense on $q_i(G^*)$, then the function $\lambda(t) \equiv 0$ on some segment $\delta \subset q_i(G^*)$ not containing points of $\Gamma_i$. By Lemma 4.3.4, there is a segment $\delta^* \subset \delta$ such that in the expression $\sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i(x, y))$ one of the terms can be deleted, without narrowing the class of the functions representable in the region $q^{-1}(\delta^*) \cap G^*$ as superpositions of the given form. Carrying out all possible deletions we can find a region $G \subset G_0 \subset D$ for which the assertion of the lemma is satisfied.

A region $G \subset D$ is called regular if, firstly, it is polyhedral and, secondly, there is a number $\gamma_G > 0$ such that for every $i > 0$ and every $t \in q_i(G)$ the set $e(q_i, t) \cap G$ is the union of a finite number of simple arcs, each of which has length not less than $\gamma_G$. A point $A$ of the boundary of the polyhedral region $G$ is called a vertex if it belongs simultaneously to two segments of the level curves of $q_i(x, y)$ and $q_j(x, y)$ $(i \neq j)$ on the boundary of $G$. Every polyhedral region has a finite number of vertices.

LEMMA 4.4.2. *For every polyhedral region $G$ and every neighbourhood $U$ of the vertices of this region we can construct a regular region $G^* \subset G$ such that $G \backslash U \subset G^*$.*

*Proof.* Let $A_1, A_2, \ldots, A_r$ be the vertices of the polyhedral region $G$; $U_1, U_2, \ldots, U_r$ suitably small neighbourhoods of these vertices. Let $k_m = k_m(A_m)$ be the number of all those functions $\{q_i(x, y)\}$ for each of which the level curve passing through the point $A_m$ does not contain any other points of the set $U_m \cap G$. Let $q_{im}(x, y)$ be one of these functions. We put $k(G) \in q_i(G)$. If $k(G) = 0$, then for any $i$ and any $t \in q_i(G)$ the length of any component of the set $e(q_i, t) \cap G$ is greater than zero and consequently the region $G$ is regular. Suppose that $k(G) > 0$ and $m$ such that $k_m \neq 0$.

We fix $\varepsilon > 0$ and put

$$G_{1m}^* = G \,\big|\, \{(x, y)\colon |q_{i_m}(x, y) - q(A_m)| < \varepsilon\} \cap U_m .$$

If $U_m$ and $\varepsilon$ are sufficiently small, then inside $U_m$ the region $G_{1m}^*$ has two vertices $A_m'$ and $A_m''$, while the region $G$ has only one vertex $A_m$ there, but $k_m(A_m') = k_m(A_m'') = k_m(A_m) - 1$. We now put $G_1^* = \cap\, G_{1m}^*$, where the intersection is taken over all $m$ such that $k_m \neq 0$. Then $k(G_1^*) = k(G) - 1$. Repeating this construction $k(G)$ times, we obtain a polyhedral region $G^*$ for which $G \backslash G^* \subset U$ and $k(G^*) = 0$. Consequently, $G^*$ is regular. This proves the lemma.

LEMMA 4.4.3. *There exists a set* $G \subset D$, *a number* $\lambda > 0$, *and a set of numbers* $\tau_i^k = 0$ *or* $1$ $(i=0, 1, \ldots, n;\ k=1, 2, \ldots, m_i)$ *such that condition* (4) *of Lemma* 4.4.1 *is satisfied, and also the conditions*

(5) *for every* $i$ *and* $t \in q_i(G)$ *and for any functions* $\{f_i^k(t)\}$

$$\max_{(x,y)\in e(q_i,t)\cap G} \left| \sum_{k=1}^{m_i} \tau_i^k p_i^k(x,y) f_i^k(q_i(x,y)) \right| \geqslant \lambda \max_k \left| \tau_i^k f_i^k(t) \right|;$$

(6) $G$ *is a regular region.*

*Proof.* By Lemma 4.4.1 there exists a region $G^* \subset D$ and a set of numbers $\tau_i^k$ such that for every polyhedral subregion $G^{**} \subset G^*$ and for every $i$ the set $\{t: \lambda(t, G^{**}, q_i, p_i^{k_1}, \ldots, p_i^{k_s}) = 0\}$ is nowhere dense in $q_i(G^{**})$, where $k_1, k_2, \ldots, k_s$ is the set of all values of $k$ for which $\tau_i^k = 1$; moreover, on the set $G^*$, for any $i$ the property (4) of Lemma 4.4.1 is satisfied. In order not to change the notation unnecessarily, we assume that all $\tau_i^k = 1$. We now construct a system of regular regions $G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_n = G$, having the following property: for every $j \leqslant i$, $\inf_{t\in q_j(G_i)} \lambda(t, G_i, q_j, \{p_j^k\}) \geqslant \lambda_i > 0$. For $G_0$ we choose any regular region $G_0 \in G^*$. Suppose that the regular regions $G_0, G_1, \ldots, G_{i-1}$ have been constructed. We now construct the set $G_i$. We denote by $\alpha_\delta$ the set $\{t: \lambda(t, q_i, G_{i-1}, \{p_i^k\}) > \delta\}$. Since the functions $\lambda(t, q_i, G_{i-1}, \{p_i^k\})$, have only finitely many points of discontinuity (of the first kind) on the set $q_i(G_{i-1})$, which consists of a finite number of segments (see Lemma 4.3.1), any component of $\alpha_\delta$ is either an interval, or a half-interval, or a segment, or a point. Suppose that the set $\alpha_\delta^N \subset \alpha_\delta$ consists of the $N$ longest components of non-zero length of the set $\alpha_\delta$ (if $\alpha_\delta$ has only $N_0$ $(<N)$ components of non-zero length, then let $\alpha_\delta^N = \alpha_\delta^{N_0}$). We denote by $\bar{\alpha}_\delta^N$ the closure of the set $\alpha_\delta^N$. We put $G_{i-1}^* = G_{i-1} \cap q_i^{-1}(\bar{\alpha}_\delta^N)$. We fix $\varepsilon > 0$. Since $G_{i-1}$ is regular, for every $j$ the length of any component of $e(q_j, t) \cap G_{i-1}$ is greater than $\gamma_G > 0$. And since the set $\{t: \lambda(t, q, G_{i-1}, \{p_i^k\}) = 0\}$ is nowhere dense in $q_i(G_{i-1})$, for sufficiently small $\delta$ and sufficiently large $N$ the set $G_{i-1}^*$ forms a $\varepsilon/2$-net on every set $e(q_j, t) \cap G_{i-1}$, $j < i$. The set $G_{i-1}^*$ is a polyhedral region. We denote by $U(\varepsilon)$ the set of points $(x, y)$ each of which is at a distance of no more than $\varepsilon/4$ from one of the vertices of the set $G_{i-1}^*$. By Lemma 4.4.2 there exists a regular region $G_i \subset G_{i-1}^*$ such that $G_{i-1}^* \backslash G_i \subset U(\varepsilon)$. The set $G_i$ forms an $\varepsilon$-net on every set $e(q_j, t) \cap G_{i-1}, j < i$ and forms an $\varepsilon/2$-net on every set $e(q_i, t) \cap G_{i-1}^*$. By Lemma 4.3.2, for sufficiently small $\varepsilon$,

$$\lambda_i = \min_{j \leq i} \inf_{t \in q_j(G_i)} \lambda(t, G_i, q_j, \{p_i^k\}) > \frac{1}{2}\min\left\{\frac{\delta}{2}, \min_{j<i} \lambda_j\right\}.$$

Thus, the regular regions $G_1, G_2, \ldots, G_n$ can be constructed. The regular region $G = G_n$ satisfies all the requirements of our lemma ($\lambda = \lambda_n$), which is now proved.

## § 5. *The set of linear superpositions in the space of continuous functions is closed*

THEOREM 4.5.1. *Suppose that continuous functions* $p_m(x, y)$ *and continuously differentiable functions* $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ *are fixed. Then in any region* $D$ *of the plane of the variables* $x, y$. *there exists a closed subregion* $G \subset D$ *such that the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

*where* $\{f_m(t)\}$ *are arbitrary continuous functions, is closed (in the uniform metric) in the set of all functions continuous on the set* $G$.

By Lemma 4.2.2 and 4.4.3 we can find a subset $G \subset D$, determine constants $\gamma > 0$ and $\lambda > 0$, and renumber the functions $\{p_m(x, y)\}$ and $\{q_m(x, y)\}$ with two indices so that the functions obtained after the renumbering, $\{p_i^k(x, y)\}$ and $\{q_i^k(x, y)\}$ $(i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i;$ $\sum_{i=0}^{n} m_i \leqslant N)$ that is, some functions may be omitted in the renumbering) satisfy conditions (1), (2), (3) of Lemma 4.2.2, and also the conditions:

(4′) for any continuous functions $\{f_m(t)\}$ there exists continuous functions $\{f_i^k(t)\}$ such that on $G$

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^k(x, y));$$

(5′) for every $i$ and $t \in q_i^1(G)$ and for any functions $\{f_i^k(t)\}$

$$\max_{(x,y) \in e(q_i^1, t) \cap G} \left| \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \right| \leqslant \lambda \max_k |f_i^k(t)|;$$

(6′) $G$ is a regular region with respect to the functions $\{q_i^k(x, y)\}$.

LEMMA 4.5.1. *In the sets* $\{ q_i^1 (G) \}$ *we can select subsets consisting of a finite number of points* $t_{i,j} \in q_i^1 (G)$ $(i=0, 1, 2, \ldots, n;\ j=1, 2, \ldots, s_i)$ *such that for any continuous functions* $\{ f_i^k (t) \}$

$$\max_{i,k} \max_{t \in q_i^1 (G)} | f_i^k (t) | \leqslant c \left( \max_{(x,y) \in G} \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k (x, y) f_i^k (q_i^1 (x, y)) \right| + \max_k | f_i^k (t_{i,j}) | \right),$$

*where* $C$ *is a constant not depending on the functions* $\{ f_i^k (t) \}$.

*Proof.* Since $G$ is polyhedral, for each $i$ we can choose in $q_i (G)$ a finite set of points $\{ t_{i,j} \}$ so dense that the components of the level curves $e (q_i^1, t_{i,j}) \cap G$ form a $\delta$-net in the set of all components of the level curves $e (q_i^1, t) \cap G$, $t \in q_i^1 (G)$. A sufficiently small $\delta$, not depending on the functions $\{ f_i^k (t) \}$, will be chosen below. We put

$$\mu = \max_{i,k} \max_{(x,y) \in G} | f_i^k (q_i^1 (x, y)) | ;$$

$$\varepsilon_1 = \max_{(x,y) \in G} \left| \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k (x, y) f_i^k (q_i^1 (x, y)) \right| ; \quad \varepsilon_2 = \max_{k,i,j} | f_i^k (t_{i,j}) | .$$

For definiteness, let $f_1^1 (q_1^1 (a)) = \mu$ at the point $a \in G$. By (5′) there exists a point $a' \in G$ such that $\left| \sum_{k=1}^{m_i} p_1^k (a') f_1^k (q_1^1 (a')) \right| \geqslant \lambda \mu$. Let $[a', a^*]$ be a segment of the level curve of the function $q_1^1 (x, y)$ with end-points at $a'$ and $a^*$ such that $h_1 ([a', a^*]) \geqslant \gamma G/2$ (see the definition of a regular region in § 4). On the arc $[a', a^*]$ we fix a point $a''$ such that $\omega (\alpha) \leqslant \dfrac{\lambda}{2m_1}$, where $\alpha = h_1 ([a', a''])$. Then on the segment $[a', a'']$ the function $\varphi_1 (x, y) = \sum_{k=1}^{m_1} p_1^k (x, y) f_1^k (q_1^1 (x, y))$ keeps'a constant sign and satisfies the inequality $| \varphi_1 (x, y) | \geqslant \lambda \mu / 2$. In fact, $| \varphi_1 (a') | \geqslant \lambda \mu$ at the point $a'$, and for any point $s \in [a', a'']$

$$| \varphi_1 (s) - \varphi_1 (a') | = \left| \sum_{k=1}^{m_1} (p_1^k (s) - p_1^k (a')) f_1^k (a') \right| \leqslant m_1 \mu \omega (\alpha) \leqslant \frac{\lambda \mu}{2}.$$

Consequently,

$$\left| \int_{s \in [a', a'']} \varphi_1 (s) \, ds \right| \geqslant \frac{1}{2} \lambda \mu \alpha .$$

By construction there is an index $j$ and a segment $[b', b'']$ of the level curve $e (q_1', t_{1,j}) \cap G$ such that $\rho ([a', a''], [b', b'']) < \delta$. We have

$$\Big| \int_{s\in[b',\,b'']} \varphi_1(s)\,ds \Big| \leqslant c_1 \varepsilon_2 \beta,$$

where $\beta = h_1([b', b''])$, $C_1 = m_1 \max_k \max_{(x,y)\in G} |p_1^k(x, y)|$. And since $\alpha$ and $\beta$ are commensurable ($\delta$ will be chosen small in comparison with $\alpha$),

$$\Big| \int_{s\in[a',\,a'']} \varphi_1(s)\,ds - \int_{s\in[b',\,b'']} \varphi_1(s)\,ds \Big| \geqslant \frac{1}{2}\lambda\mu\alpha - c_1' \varepsilon_2 \alpha.$$

By Lemma 4.2.3

$$\Big| \int_{s\in[a',\,a'']} \varphi_1(s)\,ds - \int_{s\in[b'b'']} \varphi_1(s)\,ds \Big| \leqslant c_3(\alpha\varepsilon_1 + \mu\alpha\omega(\delta) + \mu\delta).$$

Thus, $c_3(\alpha\varepsilon_1 + \mu\alpha\omega(\delta) + \mu\delta) \geqslant \lambda\mu\alpha/2 - c_1'\alpha\cdot\varepsilon_2$. If $\delta$ is taken sufficiently small in comparison with $\alpha$ (in order that $c_3(\alpha\omega(\delta) + \delta) < \lambda\alpha/2$), then we have $\mu \leqslant C(\varepsilon_1 + \varepsilon_2)$. This proves the lemma.

Let $B$ be the Banach space consisting of all systems of functions $\{f_i^k(t)\}$, defined and continuous on the sets $\{q_i^1(G)\}$, with the norm

$$\|\{f_i^k(t)\}\|_B = \max_{i,k} \max_{t\in q_i^1(G)} |f_i^k(t)| \quad (i = 0, 1, 2, \ldots, n;\ k = 1, 2, \ldots, m_i).$$

We denote by $C(G)$ the space of all functions $f(x, y)$ continuous on $G$ with the uniform metric:

$$\|f(x, y)\|_{C(G)} = \max_{(x,y)\in G} |f(x, y)|.$$

LEMMA 4.5.2. *The linear operator* $T: B \to C(G)$ *acting by the formula*

$$T(\{f_i^k(t)\}) = f(x, y) = \sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)),$$

*maps bounded closed sets of* $B$ *onto closed sets of* $C(G)$.

*Proof.* Let $F \subset B$ be a closed and bounded set of elements of $B$. Suppose that $f_n(x, y)$ is a sequence of functions in $T(F) \subset C(G)$, and that $f(x, y) \in C(G)$, where $\|f(x, y) - f_n(x, y)\|_{C(G)} \to 0$ as $n \to \infty$. We show that then $f(x, y) \in T(F)$. Since $f_n(x, y) \in T(F)$, there exists a sequence of elements $\{f_{i,n}^k(t)\} \in F$ such that $T(\{f_{i,n}^k(t)\}) = f_n(x, y)$. By Lemma 4.5.1 we can select in the sets $\{q_i^1(G)\}$ subsets consisting of a finite number of points $t_{i,j} \in q_i'(G)$ $(i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, s_i)$ such that for each element $\{f_i^k(t)\} \in B$ the inequality

$$\|\{f_i^k(t)\}\|_B \leqslant c\big(\|f(x, y)\|_{C(G)} + \max_{k,j,i} |f_i^k(t_{i,j})|\big),$$

is satisfied, where the constant $C$ does not depend on the functions $\{f^k_i(t)\}$. Since $F$ is a bounded set, there exists a subsequence of suffixes $n_1, n_2, \ldots$ such that for any $i = 0, 1, \ldots, n$; $k = 1, 2, \ldots, m_i$; $j = 1, 2, \ldots, s_i$ the numerical sequence $f^k_{i,n_\nu} \to C_{k,i,j}$ as $\nu \to \infty$. From this and the previous inequality it follows that $\{f^k_{i,n_\nu}(t)\} \in F$ $(\nu = 1, 2, \ldots)$ is a Cauchy sequence, because it is known that the sequence $f_n(x, y) \in T(F)$ is Cauchy sequence. Consequently there exists an element $\{f^k_i(t)\} \in B$ such that $\| \{f^k_i(t) - f^k_{i,n_\nu}(t)\} \|_B \to 0$. Since $F$ is a closed set, $\{f^k_i(t)\} \in F$. The operator $T: B \to C(G)$ is bounded. Therefore $T(\{f^k_i(t)\}) = f(x, y)$. Consequently $f(x, y) \in T(F)$. This proves the lemma.

The following lemma from the theory of linear operators [28] turns out to be useful.

LEMMA 4.5.3. *Let $B_1$ and $B_2$ be Banach spaces. If a linear operator $T: B_1 \to B_2$ maps bounded closed sets of $B_1$ onto closed sets of $B_2$, then its domain of values is closed.*

*Proof of Theorem* 4.5.1. The set of superpositions of the form $\sum_{m=1}^{N} p_m(x, y) f_m(g_m(x, y))$ coincides on $G$ with the set of superpositions of the form $\sum_{i=0}^{n} \sum_{k=1}^{m_i} p^k_i(x, y) f^k_i(q^1_i(x, y))$. By Lemma 4.5.2 and 4.5.3 the set of the latter superpositions is closed in the space $C(G)$. This proves the theorem.

## § 6. *The set of linear superpositions in the space of continuous functions is nowhere dense*

THEOREM 4.6.1. *For any continuous functions $p_m(x, y)$ and continuously differentiable functions $q_m(x, y)$ $(m = 1, 2, \ldots, N)$ and any region $D$ of the plane of the variables $x, y$ the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y)),$$

*where $\{f_m(t)\}$ are arbitrary continuous functions, is nowhere dense in the space of all functions continuous in $D$ with uniform convergence.*

By Lemma 4.2.2 we can find a subregion $G^* \subset D$, determine a constant $\gamma^* > 0$, and renumber the functions $\{q_m(x, y)\}$, with two indices so that the functions $\tilde{q}_i^k(x, y)$ $(i=0, 1, 2, ..., \tilde{n}; k=1, 2, ..., \tilde{m}_i; \sum_{i=0}^{\tilde{n}} \tilde{m}_i = N)$ obtained after the renumbering satisfy conditions (1), (2), (3) of Lemma 4.2.2. We now fix the point $(x_0, y_0) \in G^*$ and the number $v$ so that the line $(y-y_0) + v(x-x_0) = 0$ does not touch at any of the level curves of the functions $\tilde{q}_i^k(x, y)$ $(i=1, 2, ..., \tilde{n})$ that pass through $(x_0, y_0)$. Let $G^{**} \subset G^*$ be a disc with centre at $(x_0, y_0)$ and radius small enough so that the $\{\tilde{q}_i^k(x, y)\}$ and $q_{N+1}(x, y) = y + vx$ satisfy condition (3) of Lemma 4.2.2 with some constant $\gamma^{**} > 0$. We put $p_{N+1}(x, y) = 1$. By Lemma 4.4.3 we can find a set $G \subset G^{**}$, determine a constant $\lambda > 0$, and again renumber the functions $p_m(x, y)$ and $q_m(x, y)$ $(m=1, 2, ..., N+1)$ with two indices so that the functions $p_i^k(x, y)$ and

$$q_i^k(x, y) \quad (i=0, 1, 2, ..., n+1 \, ; \; k=1, 2, ..., m_i \, ; \; \sum_{i=0}^{n+1} m_i \leqslant N+1)$$

that is, some functions may be omitted in the renumbering) obtained after the renumbering satisfy conditions (1)-(3) of Lemma 4.2.2, conditions (4′)-(6′) of § 5, and the condition

$$7 \quad m_{n+1} = 1 \, , \; p_{N+1}^1 = p_{N+1}(x, y) = 1 \, , \; q_{N+1}^1 = q_{N+1}(x, y) = y + vx \, .$$

Let $L$ be the linear space consisting of all system of functions $\{f_i^k(t)\}$ defined and continuous on the sets $\{q_i^1(G)\}$ and satisfying the condition

$$\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x, y) f_i^k(q_i^1(x, y)) \equiv 0 \quad \text{in} \quad G \, .$$

LEMMA 4.6.1. *$L$ is a finite-dimensional linear space.*

*Proof.* By Lemma 4.5.1, in the sets $\{q_i^1(G)\}$ we can select a subset consisting of a finite number of points $\{t_{i,j}\}$ such that, if $\{f_i^k(t)\} \in L$ and $f_i^k(t_{i,j}) = 0$ for all $k, i, j$ then $f_i^k(t) \equiv 0$ on $q_i^1(G)$ for all $i, k$. Thus, the set of functions $\{f_i^k(t)\}$ is completely determined by a finite set of parameters $\{f_i^k(t_{i,j})\}$. Consequently the dimension of the space $L$ is finite. This proves the lemma.

LEMMA 4.6.2. *There exists a natural number $\mu$ such that in $D$ the polynomial $(y+vx)^\mu = Q(x, y)$ is not equal to any superposition of the form $\sum_{m=1}^{N} p_m(x, y) f_m(q_m(x, y))$, where $\{f_m(t)\}$ are arbitrary continuous functions.*

*Proof.* We denote by $\Phi$ the space of functions of the form $f(y+vx) = f^1_{n+1}(q^1_{n+1}(x,y))$ that are representable on $G$ by superpositions of the form $[\sum_{m=1}^{N} p_m(x,y) f_m(q_m(x,y))]$. Or, what comes to the same thing (see properties (4′) and (7)), of the form $[\sum_{i=0}^{n} \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^k(x,y))]$. Thus, functions of $\Phi$ satisfy the relation $\sum_{i=0}^{n+1} \sum_{k=1}^{m_i} p_i^k(x,y) f_i^k(q_i^k(x,y)) \equiv 0$ in $G$. Consequently the linear space $\Phi$ is naturally embedded in $L$. Since $L$ is finite-dimensional (Lemma 4.6.1), $\Phi$ is also finite-dimensional. Let $l$ be the dimension of $\Phi$. Since the polynomials $(y+vx), (y+vx)^2, \ldots, (y+vx)^{l+1}$ are linearly independent, at least one of them $Q(x,y) = (y+vx)^{\mu}$ is not equal to any superposition of the form under discussion on $G$ or, consequently, in $D$. This proves the lemma.

*Proof of Theorem* 4.6.1. By Lemma 4.6.2 the set of superpositions of the form given in Theorem 4.6.1 does not exhaust all continuous functions on $G$. Consequently, by Theorem 4.5.1, the set of these superpositions is a closed linear subspace of $C(G)$. Hence we conclude that the set of superpositions under discussion is nowhere dense in $C(G)$, nor consequently in $C(D)$. This proves the theorem.

COROLLARY 4.6.1. *For any continuous functions* $p_m(x_1, x_2, \ldots, x_n)$ *and continuously differentiable functions* $q_m(x_1, x_2, \ldots, x_n)$ $(m = 1, 2, \ldots, N)$ *and any region* $D$ *of the space of the variables* $(x_1, x_2, \ldots, x_n)$ *the set of superpositions of the form*

$$\sum_{m=1}^{N} p_m(x_1, x_2, \ldots, x_n) f_m(q_m(x_1, x_2, \ldots, x_n),\ x_2, x_3, , x_{n-1}),$$

*where* $\{f_m(t, x_2, x_3, \ldots, x_{n-1})\}$ *are arbitrary continuous functions of* $(n-1)$ *variables, is nowhere dense in the space of all functions continuous in* $D$ *with uniform convergence.*