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Let  $\mathcal{I}^n$  be the cube  $\{0 \leq x_i \leq 1, i = 1, \dots, n\}$ ;  $C(\mathcal{I}^n)$ -the space of all functions continuous on  $\mathcal{I}^n$  with the uniform norm;  $\Phi$ -the space of functions continuous and non-decreasing on the segment  $\mathcal{I}^1$  (with the uniform norm);  $\Phi^k = \Phi \times \dots \times \Phi$  the  $k$ -th power of the space  $\Phi$ .

THEOREM 3.2.1. *Let  $\lambda_p$  ( $p=1, \dots, n$ ) be a collection of rationally independent constants. Then for quasi every collection  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$  it is true that any function  $f \in C(\mathcal{I}^n)$  can be represented on  $\mathcal{I}^n$  in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right),$$

where  $g$  is a continuous function.

### § 3. The main lemma

We fix a function  $f \in C(\mathcal{I}^n)$ , positive numbers  $\lambda_p$  ( $p=1, \dots, n$ ) and a positive  $\varepsilon$ . We will denote by  $\Omega_f$  the set of all collections  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$  for each of which there exists a continuous function  $h$  such that  $\|h\| \leq \|f\|$  and  $\|f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)\| < (1-\varepsilon)\|f\|$ . The latter inequality is strict and consequently the set  $\Omega_f$  is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If  $\|f\| \neq 0$ , the numbers  $\{\lambda_p\}$  are rationally independent, and  $0 < \varepsilon < \frac{1}{2n+2}$ , then the corresponding set  $\Omega_f$  is everywhere dense in  $\Phi^{2n+1}$ .*

*Proof.* Let us fix an open set  $\Omega \subset \Phi^{2n+1}$  and prove that  $\Omega \cap \Omega_f$  is not empty. This will imply that  $\Omega_f$  is everywhere dense in  $\Phi^{2n+1}$ .

We choose a number  $\delta > 0$  and denote by  $\mathcal{I}_q(j)$  the segment defined by the inequality

$$q \cdot \delta + (2n+1)j \cdot \delta \leq t \leq q \cdot \delta + (2n+1)j\delta + 2n\delta$$

( $q=1, \dots, 2n+1, j$  is an integer).

The value  $\delta$  will be determined below. Now we notice, firstly, that for any  $q$  the segments  $\mathcal{I}_q(j)$  ( $j=0, \pm 1, \pm 2$ ) are pairwise disjoint and every two consecutive segments are separated by an interval of length  $\delta$  and, secondly, that, every point of the real axis belongs to at least  $2n$  of the sets  $\sum_j \mathcal{I}_q(j)$ , ( $q=1, \dots, 2n+1$ ).

We denote by  $P_q(j_1, \dots, j_n)$  the cube

$$q\delta + (2n + 1)j_k\delta \leq x_k \leq q \cdot \delta + (2n + 1)j_k\delta + 2n\delta \quad (k = 1, \dots, n).$$

We emphasise that every point  $x \in \mathcal{I}^n$  belongs to at least  $n + 1$  of the sets

$\sum_{j_1, \dots, j_n} P_q(j_1, \dots, j_n)$  ( $q = 1, \dots, 2n + 1$ ). We also remark that for any  $q$  the cubes  $\{P_q(j_1, \dots, j_n)\}$  are pairwise disjoint.

We denote by  $\Omega^*$  the subset of  $\Phi^{2n+1}$  consisting of the collections  $\varphi_1, \dots, \varphi_{2n+1}$  such that for every  $q$  the function  $\varphi_q$  is constant on every one of the segments  $\{\mathcal{I}_q(j)\}$ . We will assume that  $\delta$  is so small that  $\Omega^* \cap \Omega$  is not empty.

We choose a collection  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Omega^* \cap \Omega$ . We will show that this collection belongs to  $\Omega_f$ . We put  $t_q \equiv \sum_{p=1}^n \lambda_p \varphi_q(x_p)$ . Since the numbers  $\{\lambda_p\}$  are rationally independent we can change the constants  $\{\varphi_q(\mathcal{I}_q(j))\}$  slightly, so that the new values of  $t_q(p_q(j_1, \dots, j_n))$  are pairwise different and the collection  $\varphi_1, \dots, \varphi_{2n+1}$  remains in  $\Omega^* \cap \Omega$ .

We denote by  $f_q(j_1, \dots, j_n)$  the value of the function  $f$  at the center of  $P_q(j_1, \dots, j_n)$  and by  $h$  the function defined in the following way:

$$h(t_q(j_1, \dots, j_n)) = \frac{1}{2n + 1} f_q(j_1, \dots, j_n) \text{ outside the set } \cup_{q, j_1, \dots, j_n} t_q(j_1, \dots, j_n)$$

the function  $h$  is defined in such a way that it is continuous on the whole real

$$\text{axis and } \|h\| \leq \frac{1}{2n + 1} \|f\|.$$

Now we estimate the function  $|f - \sum_{q=1}^{2n+1} h(t_q)| = \left| \sum_{q=1}^{2n+1} \frac{f}{2n + 1} - h(t_q) \right|$ . For any  $x \in \mathcal{I}^n$ ,  $q, j_1, \dots, j_n$

$$\begin{aligned} \left| \frac{f}{2n + 1} - h(t_q) \right| &\leq \frac{1}{2n + 1} \|f\| + \|h\| \leq \frac{1}{2n + 1} \|f\| + \frac{1}{2n + 1} \|f\| \\ &= \frac{2}{2n + 1} \|f\|. \end{aligned}$$

If  $x \in P_q(j_1, \dots, j_n)$ , then

$$\begin{aligned} &\left| \frac{f}{2n + 1} - h(t_q) \right| \\ &\leq \max_{q, j_1, \dots, j_n} \left| \max_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n + 1} - \min_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n + 1} \right| = \rho. \end{aligned}$$

We recall that every  $x \in \mathcal{I}^n$  belongs to at least  $n + 1$  of the cubes  $\{P_q(j_1, \dots, j_n)\}$ . Hence

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| \leq (n+1)\rho + n \frac{2}{2n+1} \|f\|.$$

But  $\lim_{\delta \rightarrow 0} \rho = 0$ , consequently for sufficiently small  $\delta$  and  $\varepsilon < \frac{1}{2n+2}$

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| < (1 - \varepsilon) \|f\|.$$

The lemma is proved.

#### § 4. The proof of the theorem

We denote by  $F$  a countable set, everywhere dense in  $C(\mathcal{J}^n)$ . We choose  $\varepsilon$  satisfying the condition of lemma 3.3.1 and consider  $\Omega_{f_k}$  ( $f_k \in F$ ) corresponding to this  $\varepsilon$  and the collection  $\lambda_p$  mentioned in the theorem. The sets  $\{\Omega_{f_k}\}$  are open and by lemma 3.3.1 they are everywhere dense in  $\Phi^{2n+1}$ . Consequently, according to the definition, almost every element of  $\Phi^{2n+1}$  belongs to  $\Phi^* = \bigcap_{f_k \in F} \Omega_{f_k}$ .

We fix a collection  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^*$  and a function  $f \in C(\mathcal{J}^n)$  and show that the desired representation of  $f$  takes place. If  $f \equiv 0$  then as the function  $g$  we can take  $g \equiv 0$ . We will assume below that  $f \not\equiv 0$ . According to the definition of  $\Omega_{f_k}$  there exists for any  $f_k \in F$  a function  $h_k$  such that

$\left| f_k - \sum_{q=1}^{2n+1} h_k \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| \leq (1 - \varepsilon) \|f_k\|$ . The set  $F$  is everywhere dense in  $C(\mathcal{J}^n)$ . Consequently for any  $f \in C(\mathcal{J}^n)$  ( $f \not\equiv 0$ ) there exists  $h = \gamma(f)$  such that

$$\left| f - \sum_{q=1}^{2n+1} h \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| < \left( 1 - \frac{\varepsilon}{2} \right) \|f\|.$$

We define the sequence of functions  $\chi_0, \chi_1, \chi_2, \dots$  by the recurrent equalities

$$\chi_0 = f, \quad \chi_{k+1} = \chi_k - \sum_{q=1}^{2n+1} g_k \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where  $g_k = \gamma(\chi_k)$ . The series  $\sum_{k=0}^{\infty} g_k$  converges uniformly and consequently the function  $g = \sum_{k=0}^{\infty} g_k$  is continuous and

$$f - \sum_{q=1}^{2n+1} g \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) = 0.$$

The theorem is proved.