

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS
Autor: Vitushkin, A. G.
Kapitel: §3. The main lemma
DOI: <https://doi.org/10.5169/seals-48931>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 09.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Let \mathcal{I}^n be the cube $\{0 \leq x_i \leq 1, i = 1, \dots, n\}$; $C(\mathcal{I}^n)$ -the space of all functions continuous on \mathcal{I}^n with the uniform norm; Φ -the space of functions continuous and non-decreasing on the segment \mathcal{I}^1 (with the uniform norm); $\Phi^k = \Phi \times \dots \times \Phi$ the k -th power of the space Φ .

THEOREM 3.2.1. *Let $\lambda_p (p=1, \dots, n)$ be a collection of rationally independent constants. Then for quasi every collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ it is true that any function $f \in C(\mathcal{I}^n)$ can be represented on \mathcal{I}^n in the form*

$$f(x) = \sum_{q=1}^{2n+1} g \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where g is a continuous function.

§ 3. The main lemma

We fix a function $f \in C(\mathcal{I}^n)$, positive numbers $\lambda_p (p=1, \dots, n)$ and a positive ε . We will denote by Ω_f the set of all collections $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ for each of which there exists a continuous function h such that $\|h\| \leq \|f\|$ and $\|f(x) - h \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right)\| < (1-\varepsilon) \|f\|$. The latter inequality is strict and consequently the set Ω_f is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If $\|f\| \neq 0$, the numbers $\{\lambda_p\}$ are rationally independent, and $0 < \varepsilon < \frac{1}{2n+2}$, than the corresponding set Ω_f is everywhere dense in Φ^{2n+1} .*

Proof. Let us fix an open set $\Omega \subset \Phi^{2n+1}$ and prove that $\Omega \cap \Omega_f$ is not empty. This will imply that Ω_f is everywhere dense in Φ^{2n+1} .

We choose a number $\delta > 0$ and denote by $\mathcal{I}_q(j)$ the segment defined by the inequality

$$\begin{aligned} q \cdot \delta + (2n+1)j \cdot \delta &\leq t \leq q \cdot \delta + (2n+1)j\delta + 2n\delta \\ (q=1, \dots, 2n+1, j \text{ is an integer}) . \end{aligned}$$

The value δ will be determined below. Now we notice, firstly, that for any q the segments $\mathcal{I}_q(j) (j=0, \pm 1, \pm 2)$ are pairwise disjoint and every two consecutive segments are separated by an interval of length δ and, secondly, that, every point of the real axis belongs to at least $2n$ of the sets $\sum_j \mathcal{I}_q(j), (q=1, \dots, 2n+1)$.

We denote by $P_q(j_1, \dots, j_n)$ the cube

$$q\delta + (2n+1)j_k\delta \leq x_k \leq q\cdot\delta + (2n+1)j_k\delta + 2n\delta \quad (k=1, \dots, n).$$

We emphasise that every point $x \in \mathcal{I}^n$ belongs to at least $n+1$ of the sets

$\sum_{j_1, \dots, j_n} P_q(j_1, \dots, j_n)$ ($q=1, \dots, 2n+1$). We also remark that for any q the cubes $\{P_q(j_1, \dots, j_n)\}$ are pairwise disjoint.

We denote by Ω^* the subset of Φ^{2n+1} consisting of the collections $\varphi_1, \dots, \varphi_{2n+1}$ such that for every q the function φ_q is constant on every one of the segments $\{\mathcal{I}_q(j)\}$. We will assume that δ is so small that $\Omega^* \cap \Omega$ is not empty.

We choose a collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Omega^* \cap \Omega$. We will show that this collection belongs to Ω_f . We put $t_q \equiv \sum_{p=1}^n \lambda_p \varphi_q(x_p)$. Since the numbers $\{\lambda_p\}$ are rationally independent we can change the constants $\{\varphi_q(\mathcal{I}_q(j))\}$ slightly, so that the new values of $t_q(p_q(j_1, \dots, j_n))$ are pairwise different and the collection $\varphi_1, \dots, \varphi_{2n+1}$ remains in $\Omega^* \cap \Omega$.

We denote by $f_q(j_1, \dots, j_n)$ the value of the function f at the center of $P_q(j_1, \dots, j_n)$ and by h the function defined in the following way:

$h(t_q(j_1, \dots, j_n)) = \frac{1}{2n+1} f_q(j_1, \dots, j_n)$ outside the set $\cup_{q, j_1, \dots, j_n} t_q(j_1, \dots, j_n)$
the function h is defined in such a way that it is continuous on the whole real axis and $\|h\| \leq \frac{1}{2n+1} \|f\|$.

Now we estimate the function $|f - \sum_{q=1}^{2n+1} h(t_q)| = \left| \sum_{q=1}^{2n+1} \frac{f}{2n+1} - h(t_q) \right|$.
For any $x \in \mathcal{I}^n$, q, j_1, \dots, j_n

$$\begin{aligned} \left| \frac{f}{2n+1} - h(t_q) \right| &\leq \frac{1}{2n+1} \|f\| + \|h\| \leq \frac{1}{2n+1} \|f\| + \frac{1}{2n+1} \|f\| \\ &= \frac{2}{2n+1} \|f\|. \end{aligned}$$

If $x \in P_q(j_1, \dots, j_n)$, then

$$\begin{aligned} &\left| \frac{f}{2n+1} - h(t_q) \right| \\ &\leq \max_{q, j_1, \dots, j_n} \left| \max_{x \in P_q(j_1, \dots, j_n)} \frac{f(x)}{2n+1} - \min_{x \in P_q(j_1, \dots, j_n)} \frac{f(x)}{2n+1} \right| = \rho. \end{aligned}$$

We recall that every $x \in \mathcal{I}^n$ belongs to at least $n+1$ of the cubes $\{P_q(j_1, \dots, j_n)\}$. Hence

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| \leq (n+1)\rho + n \frac{2}{2n+1} \|f\|.$$

But $\lim_{\delta \rightarrow 0} \rho = 0$, consequently for sufficiently small δ and $\varepsilon < \frac{1}{2n+2}$

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| < (1-\varepsilon) \|f\|.$$

The lemma is proved.

§ 4. *The proof of the theorem*

We denote by F a countable set, everywhere dense in $C(\mathcal{I}^n)$. We choose ε satisfying the condition of lemma 3.3.1 and consider Ω_{f_k} ($f_k \in F$) corresponding to this ε and the collection λ_p mentioned in the theorem. The sets $\{\Omega_{f_k}\}$ are open and by lemma 3.3.1 they are everywhere dense in Φ^{2n+1} . Consequently, according to the definition, almost every element of Φ^{2n+1} belongs to $\Phi^* = \bigcap_{f_k \in F} \Omega_{f_k}$.

We fix a collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^*$ and a function $f \in C(\mathcal{I}^n)$ and show that the desired representation of f takes place. If $f \equiv 0$ then as the function g we can take $g \equiv 0$. We will assume below that $f \not\equiv 0$. According to the definition of Ω_{f_k} there exists for any $f_k \in F$ a function h_k such that

$\left| f_k - \sum_{q=1}^{2n+1} h_k \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| \leq (1-\varepsilon) \|f_k\|$. The set F is everywhere dense in $C(\mathcal{I}^n)$. Consequently for any $f \in C(\mathcal{I}^n)$ ($f \not\equiv 0$) there exists $h = \gamma(f)$ such that

$$\left| f - \sum_{q=1}^{2n+1} h \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| < \left(1 - \frac{\varepsilon}{2} \right) \|f\|.$$

We define the sequence of functions $\chi_0, \chi_1, \chi_2, \dots$ by the recurrent equalities

$$\chi_0 = f, \quad \chi_{k+1} = \chi_k - \sum_{q=1}^{2n+1} g_k \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where $g_k = \gamma(\chi_k)$. The series $\sum_{k=0}^{\infty} g_k$ converges uniformly and consequently the function $g = \sum_{k=0}^{\infty} g_k$ is continuous and

$$f - \sum_{q=1}^{2n+1} g \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) = 0.$$

The theorem is proved.