Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 23 (1977)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS

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Kapitel: §2. The theorem of Kahane

DOI: https://doi.org/10.5169/seals-48931

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2. The function $t_q = \sum_{p=1}^{n} \lambda_p \varphi_q(x_p)$ maps the cube $\mathcal{I}^n$ into the circle $|t| = 1$.

3. The transformation Ψ given by the equalities $t_q = \sum_{p=1}^{n} \lambda_p \varphi_q(x_p)$ $(q=1, \ldots, 2n+1)$ is one-to-one on $\mathcal{I}^n$.

4. For any function f continuous on $\mathcal{I}^n$ there exists a function $g(z)$ continuous on the disk $|z| \leqslant 1$, holomorphic inside that disk, and such that $f = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^{n} \lambda_p \varphi_q(x_p)\right)$.

The transformation Ψ gives an embedding of the cube $\mathcal{I}^n$ into the torus $|t| = 1$ $(q=1, \ldots, 2n+1)$ such that any function continuous on the cube $\tilde{\mathcal{I}}^n = \Psi(\mathcal{I}^n)$ is represented in the form $f(t_1, \ldots, t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$, where g is a function holomorphic in the unit disk. This means in particular that any function continuous on $\tilde{\mathcal{I}}^n$ has an analytic extension to the polydisk $|t_q| \leqslant 1$ $(q=1, \ldots, 2n+1)$.

§ 2. *The theorem of Kahane*

Let M be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in M. By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of M if it is true for a set of elements of second category.

Let us consider an example. Let Φ be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment $\mathcal{I}^1$ $(0 \leqslant t \leqslant 1)$. It can be shown easily that quasi every element of Φ is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as $\varphi(r') < \varphi(r'')$, where $r' < r''$ are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in Φ, and the set of all such sets is countable.

Let $\mathscr{I}^n$ be the cube $\{ 0 \leqslant x_i \leqslant 1, i = 1, ..., n \}$; $C(\mathscr{I}^n)$-the space of all functions continuous on $\mathscr{I}^n$ with the uniform norm; Φ-the space of functions continuous and non-decreasing on the segment $\mathscr{I}^1$ (with the uniform norm); $\Phi^k = \Phi \times ... \times \Phi$ the k-th power of the space Φ.

THEOREM 3.2.1. *Let λ_p $(p=1, ..., n)$ be a collection of rationally independent constants. Then for quasi every collection $\{ \varphi_1, ..., \varphi_{2n+1} \} \in \Phi^{2n+1}$ it is true that any function $f \in C(\mathscr{I}^n)$ can be represented on $\mathscr{I}^n$ in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^{n} \lambda_p \varphi_q (x_p)\right),$$

where g is a continuous function.

§ 3. *The main lemma*

We fix a function $f \in C(\mathscr{I}^n)$, positive numbers λ_p $(p=1, ..., n)$ and a positive ε. We will denote by Ω_f the set of all collections $\{ \varphi_1, ..., \varphi_{2n+1} \} \in \Phi^{2n+1}$ for each of which there exists a continuous function h such that $\| h \| \leqslant \| f \|$ and $\| f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^{n} \lambda_p \varphi_q (x_p)\right) \| < (1-\varepsilon) \| f \|$. The latter inequality is strict and consequently the set Ω_f is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If $\| f \| \neq 0$, the numbers $\{ \lambda_p \}$ are rationally independent, and $0 < \varepsilon < \dfrac{1}{2n+2}$, than the corresponding set Ω_f is everywhere dense in Φ^{2n+1}.*

Proof. Let us fix an open set $\Omega \subset \Phi^{2n+1}$ and prove that $\Omega \cap \Omega_f$ is not empty. This will imply that Ω_f is everywhere dense in Φ^{2n+1}.

We choose a number $\delta > 0$ and denote by $\mathscr{I}_q(j)$ the segment defined by the inequality

$$q \cdot \delta + (2n+1) j \cdot \delta \leqslant t \leqslant q \cdot \delta + (2n+1) j \delta + 2n\delta$$
$$(q=1, ..., 2n+1, j \text{ is an integer}).$$

The value δ will be determined below. Now we notice, firstly, that for any q the segments $\mathscr{I}_q(j)$ $(j=0, \pm 1, \pm 2)$ are pairwise desjoint and every two consecutive segments are separated by an interval of length δ and, secondly, that, every point of the real axis belongs to at least $2n$ of the sets $\sum_j \mathscr{I}_q(j)$, $(q=1, ..., 2n+1)$.