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LEMMA 2.3.3. If  $\frac{n}{s} > \frac{n'}{s'}$  then for any natural  $k$  the set  $\Omega_k \cap C_s(\mathcal{J}^n)$  is nowhere dense in  $C_s(\mathcal{J}^n)$ .

By lemma 2.3.1 and the theorem 2.2.1 for any natural  $k$   $H_\varepsilon(\Omega_k) \leq C \left(\frac{1}{\varepsilon}\right)^{n'/s'}$ , where  $C$  does not depend on  $\varepsilon$ . Hence, it follows from the inequality  $\frac{n}{s} > \frac{n'}{s'}$  and lemma 2.3.2 that the set  $\Omega_k \cap C_s(\mathcal{J}^n)$  is nowhere dense in  $C_s(\mathcal{J}^n)$ .

Now to prove the theorem we have to notice only that the set of functions from  $C_s(\mathcal{J}^n)$  representable by superpositions coincides with  $\bigcup_{k=1}^{\infty} (\Omega_k \cap C_s(\mathcal{J}^n))$ . By lemma 2.3.3 the sets  $\{\Omega_k \cap C_s(\mathcal{J}^n)\}$  are nowhere dense and consequently the set of not representable functions is a set of second category.

### CHAPTER 3. — SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

In this chapter we present the proof of the theorem of Kolmogorov given by Kahane [36]. This proof which is based on Baire's theory contains a minimum of concrete constructions and shows that there exists a wide choice of inner functions for Kolmogorov's formula.

#### § 1. *Certain improvements of Kolmogorov's theorem*

By the theorem of Kolmogorov any function defined and continuous on the cube  $\mathcal{J}^n$  can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} g_q \left( \sum_{p=1}^n \varphi_{p,q}(x_p) \right),$$

where  $\{\varphi_{p,q}\}$  are specially chosen continuous and monotonic functions which do not depend on  $f$ , and where  $\{g_q\}$  are continuous functions.

Lorentz [12] has noticed that in the theorem of Kolmogorov the functions  $\{g_q\}$  can be chosen independently of  $q$ . In fact, by adding constants to the functions  $t_q = \sum_{p=1}^n \varphi_{p,q}(x_p)$  ( $q = 1, \dots, 2n+1$ ) one can make the ranges

of the functions pairwise disjoint and consequently the functions  $\{t_q\}$  can be considered as the restrictions of a single function  $\{g_q\}$ .

Sprecher [40] has shown that the functions  $\{\varphi_{p,q}\}$  can be chosen in the form  $\varphi_{p,q}(x_p) = \lambda_p \varphi_q(x_p)$  where  $\{\lambda_p\}$  are constants and  $\{\varphi_q\}$  are continuous monotonic functions.

Thus any continuous function can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} g \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where the constants  $\{\lambda_p\}$  and the continuous monotone functions  $\{\varphi_q\}$  do not depend on  $f$ , and where  $g$  is a continuous function.

Kahane [36] has shown that such a representation is possible with almost every collection of constants  $\{\lambda_p\}$  and "quasi every" collection of continuous functions  $\{\varphi_q\}$ . The precise statement of this theorem will be given below. Here we consider some further results concerning the formula of Kolmogorov.

Doss [38] has shown that for any continuous monotonic functions  $\varphi_{p,q}$  ( $p=1, 2; q=1, 2, 3, 4$ ) there exists a continuous function  $f(x_1, x_2)$  of two variables not representable as a superposition of the form  $\sum_{q=1}^4 g_q \left( \sum_{p=1}^2 \varphi_{p,q}(x_p) \right)$ , where  $\{g_q\}$  are continuous functions.

Bassalygo [39] succeeded in showing that for any continuous functions  $\varphi_i(x_1, x_2)$  ( $i=1, 2, 3$ ) there exists a continuous function  $f(x_1, x_2)$  that is not equal to any superposition of the form  $\sum_{i=1}^3 g_i(\varphi_i(x_1, x_2))$ , where  $\{g_i\}$  are continuous functions.

Tihomirov showed that Kolmogorov's theorem can be generalized as follows: for any compact  $K$  of dimension  $n$  there exists a homeomorphic embedding  $\Psi(x) = \{\Psi_1(x), \dots, \Psi_{2n+1}(x)\}$ ,  $x \in K$  into  $(2n+1)$ -dimensional euclidean space such that any continuous function  $f(x)$  on  $K$  can be represented in the form  $f(x) = \sum_{i=1}^{2n+1} g_i(\Psi_i(x))$ , where  $\{g_i\}$  are continuous functions of one variable.

In the same paper [36] Kahane has shown that there exist complex numbers  $\lambda_p$  ( $p=1, \dots, n$ ) and complex valued functions  $\varphi_q$  ( $q=1, \dots, 2n+1$ ) possessing the following properties.

1. The function  $\varphi_q$  is a monotonic continuous transformation of the real axis onto the circle  $|t| = 1$  ( $q=1, \dots, 2n+1$ ).

2. The function  $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$  maps the cube  $\mathcal{J}^n$  into the circle  $|t| = 1$ .

3. The transformation  $\Psi$  given by the equalities  $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$  ( $q=1, \dots, 2n+1$ ) is one-to-one on  $\mathcal{J}^n$ .

4. For any function  $f$  continuous on  $\mathcal{J}^n$  there exists a function  $g(z)$  continuous on the disk  $|z| \leq 1$ , holomorphic inside that disk, and such that  $f = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)$ .

The transformation  $\Psi$  gives an embedding of the cube  $\mathcal{J}^n$  into the torus  $|t| = 1$  ( $q=1, \dots, 2n+1$ ) such that any function continuous on the cube  $\tilde{\mathcal{J}}^n = \Psi(\mathcal{J}^n)$  is represented in the form  $f(t_1, \dots, t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$ , where  $g$  is a function holomorphic in the unit disk. This means in particular that any function continuous on  $\tilde{\mathcal{J}}^n$  has an analytic extension to the polydisk  $|t_q| \leq 1$  ( $q=1, \dots, 2n+1$ ).

## § 2. The theorem of Kahane

Let  $M$  be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in  $M$ . By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of  $M$  if it is true for a set of elements of second category.

Let us consider an example. Let  $\Phi$  be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment  $\mathcal{J}^1$  ( $0 \leq t \leq 1$ ). It can be shown easily that quasi every element of  $\Phi$  is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as  $\varphi(r') < \varphi(r'')$ , where  $r' < r''$  are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in  $\Phi$ , and the set of all such sets is countable.

Let  $\mathcal{J}^n$  be the cube  $\{0 \leq x_i \leq 1, i = 1, \dots, n\}$ ;  $C(\mathcal{J}^n)$ -the space of all functions continuous on  $\mathcal{J}^n$  with the uniform norm;  $\Phi$ -the space of functions continuous and non-decreasing on the segment  $\mathcal{J}^1$  (with the uniform norm);  $\Phi^k = \Phi \times \dots \times \Phi$  the  $k$ -th power of the space  $\Phi$ .

THEOREM 3.2.1. *Let  $\lambda_p$  ( $p=1, \dots, n$ ) be a collection of rationally independent constants. Then for quasi every collection  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$  it is true that any function  $f \in C(\mathcal{J}^n)$  can be represented on  $\mathcal{J}^n$  in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right),$$

where  $g$  is a continuous function.

### § 3. The main lemma

We fix a function  $f \in C(\mathcal{J}^n)$ , positive numbers  $\lambda_p$  ( $p=1, \dots, n$ ) and a positive  $\varepsilon$ . We will denote by  $\Omega_f$  the set of all collections  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$  for each of which there exists a continuous function  $h$  such that  $\|h\| \leq \|f\|$  and  $\|f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)\| < (1-\varepsilon)\|f\|$ . The latter inequality is strict and consequently the set  $\Omega_f$  is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If  $\|f\| \neq 0$ , the numbers  $\{\lambda_p\}$  are rationally independent, and  $0 < \varepsilon < \frac{1}{2n+2}$ , then the corresponding set  $\Omega_f$  is everywhere dense in  $\Phi^{2n+1}$ .*

*Proof.* Let us fix an open set  $\Omega \subset \Phi^{2n+1}$  and prove that  $\Omega \cap \Omega_f$  is not empty. This will imply that  $\Omega_f$  is everywhere dense in  $\Phi^{2n+1}$ .

We choose a number  $\delta > 0$  and denote by  $\mathcal{J}_q(j)$  the segment defined by the inequality

$$q \cdot \delta + (2n+1)j \cdot \delta \leq t \leq q \cdot \delta + (2n+1)j\delta + 2n\delta$$

$$(q=1, \dots, 2n+1, j \text{ is an integer}).$$

The value  $\delta$  will be determined below. Now we notice, firstly, that for any  $q$  the segments  $\mathcal{J}_q(j)$  ( $j=0, \pm 1, \pm 2$ ) are pairwise disjoint and every two consecutive segments are separated by an interval of length  $\delta$  and, secondly, that, every point of the real axis belongs to at least  $2n$  of the sets  $\sum_j \mathcal{J}_q(j)$ , ( $q=1, \dots, 2n+1$ ).

We denote by  $P_q(j_1, \dots, j_n)$  the cube

$$q\delta + (2n+1)j_k\delta \leq x_k \leq q \cdot \delta + (2n+1)j_k\delta + 2n\delta \quad (k=1, \dots, n).$$

We emphasise that every point  $x \in \mathcal{J}^n$  belongs to at least  $n+1$  of the sets

$\sum_{j_1, \dots, j_n} P_q(j_1, \dots, j_n) \quad (q=1, \dots, 2n+1)$ . We also remark that for any  $q$  the cubes  $\{P_q(j_1, \dots, j_n)\}$  are pairwise disjoint.

We denote by  $\Omega^*$  the subset of  $\Phi^{2n+1}$  consisting of the collections  $\varphi_1, \dots, \varphi_{2n+1}$  such that for every  $q$  the function  $\varphi_q$  is constant on every one of the segments  $\{\mathcal{J}_q(j)\}$ . We will assume that  $\delta$  is so small that  $\Omega^* \cap \Omega$  is not empty.

We choose a collection  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Omega^* \cap \Omega$ . We will show that this collection belongs to  $\Omega_f$ . We put  $t_q \equiv \sum_{p=1}^n \lambda_p \varphi_q(x_p)$ . Since the numbers  $\{\lambda_p\}$  are rationally independent we can change the constants  $\{\varphi_q(\mathcal{J}_q(j))\}$  slightly, so that the new values of  $t_q(p_q(j_1, \dots, j_n))$  are pairwise different and the collection  $\varphi_1, \dots, \varphi_{2n+1}$  remains in  $\Omega^* \cap \Omega$ .

We denote by  $f_q(j_1, \dots, j_n)$  the value of the function  $f$  at the center of  $P_q(j_1, \dots, j_n)$  and by  $h$  the function defined in the following way:

$$h(t_q(j_1, \dots, j_n)) = \frac{1}{2n+1} f_q(j_1, \dots, j_n) \text{ outside the set } \bigcup_{q, j_1, \dots, j_n} t_q(j_1, \dots, j_n)$$

the function  $h$  is defined in such a way that it is continuous on the whole real axis and  $\|h\| \leq \frac{1}{2n+1} \|f\|$ .

Now we estimate the function  $|f - \sum_{q=1}^{2n+1} h(t_q)| = \left| \sum_{q=1}^{2n+1} \frac{f}{2n+1} - h(t_q) \right|$ . For any  $x \in \mathcal{J}^n$ ,  $q, j_1, \dots, j_n$

$$\begin{aligned} \left| \frac{f}{2n+1} - h(t_q) \right| &\leq \frac{1}{2n+1} \|f\| + \|h\| \leq \frac{1}{2n+1} \|f\| + \frac{1}{2n+1} \|f\| \\ &= \frac{2}{2n+1} \|f\|. \end{aligned}$$

If  $x \in P_q(j_1, \dots, j_n)$ , then

$$\begin{aligned} &\left| \frac{f}{2n+1} - h(t_q) \right| \\ &\leq \max_{q, j_1, \dots, j_n} \left| \max_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n+1} - \min_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n+1} \right| = \rho. \end{aligned}$$

We recall that every  $x \in \mathcal{J}^n$  belongs to at least  $n+1$  of the cubes  $\{P_q(j_1, \dots, j_n)\}$ . Hence

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| \leq (n+1)\rho + n \frac{2}{2n+1} \|f\|.$$

But  $\lim_{\delta \rightarrow 0} \rho = 0$ , consequently for sufficiently small  $\delta$  and  $\varepsilon < \frac{1}{2n+2}$

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| < (1-\varepsilon) \|f\|.$$

The lemma is proved.

#### § 4. The proof of the theorem

We denote by  $F$  a countable set, everywhere dense in  $C(\mathcal{J}^n)$ . We choose  $\varepsilon$  satisfying the condition of lemma 3.3.1 and consider  $\Omega_{f_k}$  ( $f_k \in F$ ) corresponding to this  $\varepsilon$  and the collection  $\lambda_p$  mentioned in the theorem. The sets  $\{\Omega_{f_k}\}$  are open and by lemma 3.3.1 they are everywhere dense in  $\Phi^{2n+1}$ . Consequently, according to the definition, almost every element of  $\Phi^{2n+1}$  belongs to  $\Phi^* = \bigcap_{f_k \in F} \Omega_{f_k}$ .

We fix a collection  $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^*$  and a function  $f \in C(\mathcal{J}^n)$  and show that the desired representation of  $f$  takes place. If  $f \equiv 0$  then as the function  $g$  we can take  $g \equiv 0$ . We will assume below that  $f \not\equiv 0$ . According to the definition of  $\Omega_{f_k}$  there exists for any  $f_k \in F$  a function  $h_k$  such that  $\left| f_k - \sum_{q=1}^{2n+1} h_k \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| \leq (1-\varepsilon) \|f_k\|$ . The set  $F$  is everywhere dense in  $C(\mathcal{J}^n)$ . Consequently for any  $f \in C(\mathcal{J}^n)$  ( $f \not\equiv 0$ ) there exists  $h = \gamma(f)$  such that

$$\left| f - \sum_{q=1}^{2n+1} h \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| < \left( 1 - \frac{\varepsilon}{2} \right) \|f\|.$$

We define the sequence of functions  $\chi_0, \chi_1, \chi_2, \dots$  by the recurrent equalities

$$\chi_0 = f, \quad \chi_{k+1} = \chi_k - \sum_{q=1}^{2n+1} g_k \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where  $g_k = \gamma(\chi_k)$ . The series  $\sum_{k=0}^{\infty} g_k$  converges uniformly and consequently the function  $g = \sum_{k=0}^{\infty} g_k$  is continuous and

$$f - \sum_{q=1}^{2n+1} g \left( \sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) = 0.$$

The theorem is proved.