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LEMMA 2.3.3. If $\frac{n}{s} > \frac{n'}{s'}$ then for any natural k the set $\Omega_k \cap C_s(\mathcal{J}^n)$ is nowhere dense in $C_s(\mathcal{J}^n)$.

By lemma 2.3.1 and the theorem 2.2.1 for any natural k $H_\varepsilon(\Omega_k) \leq C \left(\frac{1}{\varepsilon}\right)^{n'/s'}$, where C does not depend on ε . Hence, it follows from the inequality $\frac{n}{s} > \frac{n'}{s'}$ and lemma 2.3.2 that the set $\Omega_k \cap C_s(\mathcal{J}^n)$ is nowhere dense in $C_s(\mathcal{J}^n)$.

Now to prove the theorem we have to notice only that the set of functions from $C_s(\mathcal{J}^n)$ representable by superpositions coincides with $\bigcup_{k=1}^{\infty} (\Omega_k \cap C_s(\mathcal{J}^n))$. By lemma 2.3.3 the sets $\{\Omega_k \cap C_s(\mathcal{J}^n)\}$ are nowhere dense and consequently the set of not representable functions is a set of second category.

CHAPTER 3. — SUPERPOSITIONS OF CONTINUOUS FUNCTIONS

In this chapter we present the proof of the theorem of Kolmogorov given by Kahane [36]. This proof which is based on Baire's theory contains a minimum of concrete constructions and shows that there exists a wide choice of inner functions for Kolmogorov's formula.

§ 1. *Certain improvements of Kolmogorov's theorem*

By the theorem of Kolmogorov any function defined and continuous on the cube \mathcal{J}^n can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} g_q \left(\sum_{p=1}^n \varphi_{p,q}(x_p) \right),$$

where $\{\varphi_{p,q}\}$ are specially chosen continuous and monotonic functions which do not depend on f , and where $\{g_q\}$ are continuous functions.

Lorentz [12] has noticed that in the theorem of Kolmogorov the functions $\{g_q\}$ can be chosen independently of q . In fact, by adding constants to the functions $t_q = \sum_{p=1}^n \varphi_{p,q}(x_p)$ ($q = 1, \dots, 2n+1$) one can make the ranges

of the functions pairwise disjoint and consequently the functions $\{t_q\}$ can be considered as the restrictions of a single function $\{g_q\}$.

Sprecher [40] has shown that the functions $\{\varphi_{p,q}\}$ can be chosen in the form $\varphi_{p,q}(x_p) = \lambda_p \varphi_q(x_p)$ where $\{\lambda_p\}$ are constants and $\{\varphi_q\}$ are continuous monotonic functions.

Thus any continuous function can be represented as

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right),$$

where the constants $\{\lambda_p\}$ and the continuous monotone functions $\{\varphi_q\}$ do not depend on f , and where g is a continuous function.

Kahane [36] has shown that such a representation is possible with almost every collection of constants $\{\lambda_p\}$ and "quasi every" collection of continuous functions $\{\varphi_q\}$. The precise statement of this theorem will be given below. Here we consider some further results concerning the formula of Kolmogorov.

Doss [38] has shown that for any continuous monotonic functions $\varphi_{p,q}$ ($p=1, 2; q=1, 2, 3, 4$) there exists a continuous function $f(x_1, x_2)$ of two variables not representable as a superposition of the form $\sum_{q=1}^4 g_q\left(\sum_{p=1}^2 \varphi_{p,q}(x_p)\right)$, where $\{g_q\}$ are continuous functions.

Bassalygo [39] succeeded in showing that for any continuous functions $\varphi_i(x_1, x_2)$ ($i=1, 2, 3$) there exists a continuous function $f(x_1, x_2)$ that is not equal to any superposition of the form $\sum_{i=1}^3 g_i(\varphi_i(x_1, x_2))$, where $\{g_i\}$ are continuous functions.

Tihomirov showed that Kolmogorov's theorem can be generalized as follows: for any compact K of dimension n there exists a homeomorphic embedding $\Psi(x) = \{\Psi_1(x), \dots, \Psi_{2n+1}(x)\}$, $x \in K$ into $(2n+1)$ -dimensional euclidean space such that any continuous function $f(x)$ on K can be represented in the form $f(x) = \sum_{i=1}^{2n+1} g_i(\Psi_i(x))$, where $\{g_i\}$ are continuous functions of one variable.

In the same paper [36] Kahane has shown that there exist complex numbers λ_p ($p=1, \dots, n$) and complex valued functions φ_q ($q=1, \dots, 2n+1$) possessing the following properties.

1. The function φ_q is a monotonic continuous transformation of the real axis onto the circle $|t| = 1$ ($q=1, \dots, 2n+1$).

2. The function $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$ maps the cube \mathcal{J}^n into the circle $|t| = 1$.

3. The transformation Ψ given by the equalities $t_q = \sum_{p=1}^n \lambda_p \varphi_q(x_p)$ ($q=1, \dots, 2n+1$) is one-to-one on \mathcal{J}^n .

4. For any function f continuous on \mathcal{J}^n there exists a function $g(z)$ continuous on the disk $|z| \leq 1$, holomorphic inside that disk, and such that $f = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)$.

The transformation Ψ gives an embedding of the cube \mathcal{J}^n into the torus $|t| = 1$ ($q=1, \dots, 2n+1$) such that any function continuous on the cube $\tilde{\mathcal{J}}^n = \Psi(\mathcal{J}^n)$ is represented in the form $f(t_1, \dots, t_{2n+1}) = \sum_{q=1}^{2n+1} g(t_q)$, where g is a function holomorphic in the unit disk. This means in particular that any function continuous on $\tilde{\mathcal{J}}^n$ has an analytic extension to the polydisk $|t_q| \leq 1$ ($q=1, \dots, 2n+1$).

§ 2. The theorem of Kahane

Let M be a complete metric space. We recall that a set is called a set of second category if it is the intersection of a countable family of open sets which are everywhere dense in M . By the theorem of Baire in a complete metric space no set of second category is empty. The massivity of such sets is characterized by the fact that the intersection of a countable family of sets of second category is again a set of second category and consequently is not empty.

We will say that a statement is true for quasi every element of M if it is true for a set of elements of second category.

Let us consider an example. Let Φ be the space with uniform norm consisting of all functions continuous and non-decreasing on the segment \mathcal{J}^1 ($0 \leq t \leq 1$). It can be shown easily that quasi every element of Φ is a strictly increasing function.

In fact, any strictly increasing function belongs to any set defined as $\varphi(r') < \varphi(r'')$, where $r' < r''$ are fixed rational numbers. Any set defined by an inequality of that type is open and everywhere dense in Φ , and the set of all such sets is countable.

Let \mathcal{J}^n be the cube $\{0 \leq x_i \leq 1, i = 1, \dots, n\}$; $C(\mathcal{J}^n)$ -the space of all functions continuous on \mathcal{J}^n with the uniform norm; Φ -the space of functions continuous and non-decreasing on the segment \mathcal{J}^1 (with the uniform norm); $\Phi^k = \Phi \times \dots \times \Phi$ the k -th power of the space Φ .

THEOREM 3.2.1. *Let λ_p ($p=1, \dots, n$) be a collection of rationally independent constants. Then for quasi every collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ it is true that any function $f \in C(\mathcal{J}^n)$ can be represented on \mathcal{J}^n in the form*

$$f(x) = \sum_{q=1}^{2n+1} g\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right),$$

where g is a continuous function.

§ 3. The main lemma

We fix a function $f \in C(\mathcal{J}^n)$, positive numbers λ_p ($p=1, \dots, n$) and a positive ε . We will denote by Ω_f the set of all collections $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^{2n+1}$ for each of which there exists a continuous function h such that $\|h\| \leq \|f\|$ and $\|f(x) - \sum_{q=1}^{2n+1} h\left(\sum_{p=1}^n \lambda_p \varphi_q(x_p)\right)\| < (1-\varepsilon)\|f\|$. The latter inequality is strict and consequently the set Ω_f is open.

The idea of the construction is contained in the following statement.

LEMMA 3.3.1. *If $\|f\| \neq 0$, the numbers $\{\lambda_p\}$ are rationally independent, and $0 < \varepsilon < \frac{1}{2n+2}$, then the corresponding set Ω_f is everywhere dense in Φ^{2n+1} .*

Proof. Let us fix an open set $\Omega \subset \Phi^{2n+1}$ and prove that $\Omega \cap \Omega_f$ is not empty. This will imply that Ω_f is everywhere dense in Φ^{2n+1} .

We choose a number $\delta > 0$ and denote by $\mathcal{J}_q(j)$ the segment defined by the inequality

$$q \cdot \delta + (2n+1)j \cdot \delta \leq t \leq q \cdot \delta + (2n+1)j\delta + 2n\delta$$

$$(q=1, \dots, 2n+1, j \text{ is an integer}).$$

The value δ will be determined below. Now we notice, firstly, that for any q the segments $\mathcal{J}_q(j)$ ($j=0, \pm 1, \pm 2$) are pairwise disjoint and every two consecutive segments are separated by an interval of length δ and, secondly, that, every point of the real axis belongs to at least $2n$ of the sets $\sum_j \mathcal{J}_q(j)$, ($q=1, \dots, 2n+1$).

We denote by $P_q(j_1, \dots, j_n)$ the cube

$$q\delta + (2n+1)j_k\delta \leq x_k \leq q \cdot \delta + (2n+1)j_k\delta + 2n\delta \quad (k=1, \dots, n).$$

We emphasise that every point $x \in \mathcal{J}^n$ belongs to at least $n+1$ of the sets

$\sum_{j_1, \dots, j_n} P_q(j_1, \dots, j_n) \quad (q=1, \dots, 2n+1)$. We also remark that for any q the cubes $\{P_q(j_1, \dots, j_n)\}$ are pairwise disjoint.

We denote by Ω^* the subset of Φ^{2n+1} consisting of the collections $\varphi_1, \dots, \varphi_{2n+1}$ such that for every q the function φ_q is constant on every one of the segments $\{\mathcal{J}_q(j)\}$. We will assume that δ is so small that $\Omega^* \cap \Omega$ is not empty.

We choose a collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Omega^* \cap \Omega$. We will show that this collection belongs to Ω_f . We put $t_q \equiv \sum_{p=1}^n \lambda_p \varphi_q(x_p)$. Since the numbers $\{\lambda_p\}$ are rationally independent we can change the constants $\{\varphi_q(\mathcal{J}_q(j))\}$ slightly, so that the new values of $t_q(p_q(j_1, \dots, j_n))$ are pairwise different and the collection $\varphi_1, \dots, \varphi_{2n+1}$ remains in $\Omega^* \cap \Omega$.

We denote by $f_q(j_1, \dots, j_n)$ the value of the function f at the center of $P_q(j_1, \dots, j_n)$ and by h the function defined in the following way:

$$h(t_q(j_1, \dots, j_n)) = \frac{1}{2n+1} f_q(j_1, \dots, j_n) \text{ outside the set } \bigcup_{q, j_1, \dots, j_n} t_q(j_1, \dots, j_n)$$

the function h is defined in such a way that it is continuous on the whole real axis and $\|h\| \leq \frac{1}{2n+1} \|f\|$.

Now we estimate the function $|f - \sum_{q=1}^{2n+1} h(t_q)| = \left| \sum_{q=1}^{2n+1} \frac{f}{2n+1} - h(t_q) \right|$. For any $x \in \mathcal{J}^n$, q, j_1, \dots, j_n

$$\begin{aligned} \left| \frac{f}{2n+1} - h(t_q) \right| &\leq \frac{1}{2n+1} \|f\| + \|h\| \leq \frac{1}{2n+1} \|f\| + \frac{1}{2n+1} \|f\| \\ &= \frac{2}{2n+1} \|f\|. \end{aligned}$$

If $x \in P_q(j_1, \dots, j_n)$, then

$$\begin{aligned} &\left| \frac{f}{2n+1} - h(t_q) \right| \\ &\leq \max_{q, j_1, \dots, j_n} \left| \max_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n+1} - \min_{x \in p_q(j_1, \dots, j_n)} \frac{f(x)}{2n+1} \right| = \rho. \end{aligned}$$

We recall that every $x \in \mathcal{J}^n$ belongs to at least $n+1$ of the cubes $\{P_q(j_1, \dots, j_n)\}$. Hence

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| \leq (n+1)\rho + n \frac{2}{2n+1} \|f\|.$$

But $\lim_{\delta \rightarrow 0} \rho = 0$, consequently for sufficiently small δ and $\varepsilon < \frac{1}{2n+2}$

$$\left| f - \sum_{q=1}^{2n+1} h(t_q) \right| < (1-\varepsilon) \|f\|.$$

The lemma is proved.

§ 4. The proof of the theorem

We denote by F a countable set, everywhere dense in $C(\mathcal{J}^n)$. We choose ε satisfying the condition of lemma 3.3.1 and consider Ω_{f_k} ($f_k \in F$) corresponding to this ε and the collection λ_p mentioned in the theorem. The sets $\{\Omega_{f_k}\}$ are open and by lemma 3.3.1 they are everywhere dense in Φ^{2n+1} . Consequently, according to the definition, almost every element of Φ^{2n+1} belongs to $\Phi^* = \bigcap_{f_k \in F} \Omega_{f_k}$.

We fix a collection $\{\varphi_1, \dots, \varphi_{2n+1}\} \in \Phi^*$ and a function $f \in C(\mathcal{J}^n)$ and show that the desired representation of f takes place. If $f \equiv 0$ then as the function g we can take $g \equiv 0$. We will assume below that $f \not\equiv 0$. According to the definition of Ω_{f_k} there exists for any $f_k \in F$ a function h_k such that $\left| f_k - \sum_{q=1}^{2n+1} h_k \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| \leq (1-\varepsilon) \|f_k\|$. The set F is everywhere dense in $C(\mathcal{J}^n)$. Consequently for any $f \in C(\mathcal{J}^n)$ ($f \not\equiv 0$) there exists $h = \gamma(f)$ such that

$$\left| f - \sum_{q=1}^{2n+1} h \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) \right| < \left(1 - \frac{\varepsilon}{2} \right) \|f\|.$$

We define the sequence of functions $\chi_0, \chi_1, \chi_2, \dots$ by the recurrent equalities

$$\chi_0 = f, \quad \chi_{k+1} = \chi_k - \sum_{q=1}^{2n+1} g_k \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right),$$

where $g_k = \gamma(\chi_k)$. The series $\sum_{k=0}^{\infty} g_k$ converges uniformly and consequently the function $g = \sum_{k=0}^{\infty} g_k$ is continuous and

$$f - \sum_{q=1}^{2n+1} g \left(\sum_{p=1}^n \lambda_p \varphi_q(x_p) \right) = 0.$$

The theorem is proved.