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Autor: Vitushkin, A. G.
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$$H_\varepsilon(F_{\rho_1, \rho_2, \dots, \rho_n}^c) = \frac{1}{(n+1)!} \prod_{k=1}^n \frac{1}{\log \rho_k} \left(\log \frac{c}{\varepsilon}\right)^{n+1} + O\left[\left(\log \frac{c}{\varepsilon}\right)^n \log \log \frac{c}{\varepsilon}\right].$$

3. Let $F_{s,c}^n$ be the class of real valued functions on the cube $\{-1 \leq x_k \leq 1\}$ ($k=1, \dots, n$), bounded in modulus on that cube by the constant s_k and such that their analytic extensions are entire functions of order s_k with respect to $z_k = x_k + iy_k$ ($k=1, \dots, n$). Then

$$\begin{aligned} H_\varepsilon(F_{s,c}^n) &= \frac{1}{(n+1)!} \prod_{k=1}^n s_k \left(\log \frac{c}{\varepsilon}\right)^{n+1} \left(\log \log \frac{c}{\varepsilon}\right)^{-n} + \\ &= O\left[\left(\log \frac{c}{\varepsilon}\right)^{n+1} \left(\log \log \frac{c}{\varepsilon}\right)^{-n-1}\right]. \end{aligned}$$

These estimates and other results connected with estimates of entropy and applications are to be found for example in [49]-[53].

§ 2. The entropy of the space of smooth functions

Here we give an estimate of the entropy of the class of S times differentiable functions of n variables. The lower estimate was obtained in [4], the upper one—in [23].

We fix integers $n \geq 1$ and $p \geq 0$ and numbers $0 \leq \alpha \leq 1$, $L > 0$, $C > 0$, $\rho > 0$. We will denote by \mathcal{I} the cube $0 \leq x_i \leq \rho$ ($i=1, \dots, n$) and by $F = F_{S,L,c}^{\rho,n}$ ($S=p+\alpha$) the set of all real valued functions defined on \mathcal{I} such that their partial derivatives of order p satisfy the condition Lip α with the constant L and

$$\left| \frac{\partial^{k_1+\dots+k_n} f(0)}{\partial^{k_1} x_1 \dots \partial^{k_n} x_n} \right| \leq c \quad \left(\sum_{i=1}^n k_i \leq p \right)$$

We say that the function $g(x)$ satisfies the condition Lip α with the constant L if for any x' and x''

$$|g(x') - g(x'')| \leq L(r(x', x''))^\alpha,$$

where $r(x', x'')$ is the distance between x' and x'' .

THEOREM 2.2.1. *If $\varepsilon > 0$ is sufficiently small then*

$$A\rho^n \left(\frac{L}{\varepsilon}\right)^{n/s} \leq H_\varepsilon(F) \leq B\rho^n \left(\frac{L}{\varepsilon}\right)^{n/s},$$

where A and B are positive constants depending only on s and n .

We choose $\delta > 0$ such that the number ρ/δ is an integer. We divide the cube \mathcal{J} into $\left(\frac{\rho}{\delta}\right)^n$ cubes P_i ($i = 1, 2, \dots, \left(\frac{\rho}{\delta}\right)^n$) by hyperplanes, parallel to its $(n-1)$ -dimensional edges. Each of the cubes P_i has side of length δ , and the edges of these cubes are parallel to those on \mathcal{J} . Let C_i denote the centre of the cube P_i and S_i the n -dimensional closed sphere (inscribed in P_i) of radius $\delta/2$ and centre at the point C_i . Put

$$\varphi_i(x) = \varphi_i(x_1, x_2, \dots, x_n) = \begin{cases} 0, & \text{if } x \in \mathcal{J} - S_i \\ A \left(1 + \cos \left(\frac{2\pi}{\delta} r(C_i, x) \right) \right)^p & \text{if } x \in S_i, \end{cases}$$

where $r(C_i, x)$ is the distance from the point x to the centre C_i of the sphere S_i . Put, further,

$$\varphi_{\eta_1, \eta_2, \dots, \eta_h}(x) = \sum_{i=1}^h \eta_i \varphi_i(x) \\ \left(\eta_i = \pm 1; i = 1, 2, \dots, h; h = \left(\frac{\rho}{\delta}\right)^n \right).$$

LEMMA 2.2.1. *We can find a positive number $A(s, L, n)$, such that when $A = A(s, L, n) \delta^s$ and given any set of numbers η_i ($i = 1, 2, \dots, h$)-the corresponding function $\varphi_{\eta_1, \eta_2, \dots, \eta_h}(x)$ belongs to F .*

Proof. By differentiating $\varphi_i(x)$ it is not difficult to see that inside the sphere S_i its partial derivatives of all orders exist. And the modulus of any partial derivative of order k is bounded inside S_i by $AB(s, k, n) \delta^{-k}$, where $B(s, k, n)$ is some constant, depending only on s, k, n . In particular, any derivative of the function $\varphi_i(x)$ of order $p + 1$ is bounded in the sphere S_i by the constant

$$AB(s, p + 1, n) \delta^{-p-1} = \frac{A(s, L, n) B(s, p + 1, n)}{\delta^{1-\alpha}}.$$

Let $g(x)$ be any p -th order partial derivative of the functions $\varphi_i(x)$. We take two points a and b belonging to the sphere S_i . Then $g(b) - g(a) = r(a, b) \frac{\partial g(c)}{\partial r}$, where $\frac{\partial g(c)}{\partial r}$ is the derivative of $g(x)$ along the direction (a, b) , taken at some point c of $[a, b]$. Since any $p + 1$ -th order partial derivative of $\varphi_i(x)$ is bounded inside the sphere by the constant

$$\frac{A(s, L, n) B(s, p + 1, n)}{\delta^{1-\alpha}}, \text{ we have } \left| \frac{\partial g(c)}{\partial r} \right| \leq n \frac{A(s, L, n) B(s, p + 1, n)}{\delta^{1-\alpha}}$$

And then

$$|g(b) - g(a)| \leq \left| \rho \frac{\partial g(c)}{\partial r} \right| \leq \rho n \frac{A(s, L, n) B(s, p+1, n)}{\delta^{1-\alpha}} \\ \leq \rho^\alpha n A(s, L, n) B(s, p+1, n).$$

Put

$$A(s, L, n) = \frac{L}{2n B(s, p+1, n)}.$$

Then

$$|g(b) - g(a)| \leq \frac{1}{2} L \rho^\alpha.$$

Now let $\Psi(x)$ be any of the p -th partial derivatives of the function $\varphi_{\eta_1, \eta_2, \dots, \eta_h}(x)$. We choose two points x' and x'' of \mathcal{J} ($x' \in S_i, x'' \in S_j$) and let $g_1(x)$ and $g_2(x)$ be the partial derivatives of the same kind as $\Psi(x)$ of the functions $\varphi_i(x)$ and $\varphi_j(x)$ (respectively). It is easy to verify that $g_1(x)$ and $g_2(x)$ are continuous on \mathcal{J} and identically equal to zero on the sets $\mathcal{J} - S_i$ and $\mathcal{J} - S_j$ (respectively). We select some point x_0 belonging to the boundary of the sphere S_i and lying on the segment $[x', x'']$.

Then

$$|\psi(x'') - \psi(x')| \leq |g_1(x'') - g_1(x')| + |g_2(x'') - g_2(x')| \\ \leq |g_1(x') - g_1(x_0)| + |g_2(x'') - g_2(x_0)| \leq |g(b) - g(a)| \\ \leq \frac{1}{2} L (r(x', x_0))^\alpha + \frac{1}{2} L (r(x'', x_0))^\alpha \leq L (r(x', x''))^\alpha.$$

If one of the points x', x'' (or both) belongs to the set $\mathcal{J} - \sum_{i=1}^h S_i$, then we can prove similarly that

$$|\varphi(x'') - \varphi(x')| \leq L (r(x', x''))^\alpha.$$

Q.E.D.

LEMMA 2.2.2. *There exists a positive constant A , depending only on s, L, n such that for sufficiently small ε*

$$H_\varepsilon(F) \geq A \rho^n \left(\frac{1}{\varepsilon} \right)^{n/s}.$$

Proof. We choose some positive number $k > 1$ such that when $\delta = \left(\frac{k\varepsilon}{A(s, L, n)} \right)^{1/s}$ is an integer.

We choose two different functions of the type $\varphi_{\eta_1, \dots, \eta_h}(x)$ and $\varphi_{\tau_1, \tau_2, \dots, \tau_h}(x)$, $A = A(s, L, n) \delta^s$ and $A(s, L, n)$ is taken so small that both functions belong to the family F . Since the functions we have chosen are assumed to be different, for some i $\tau_i \neq \eta_i$. And therefore

$$\begin{aligned} & | \varphi_{\eta_1, \eta_2, \dots, \eta_h}(c_i) - \varphi_{\tau_1, \tau_2, \dots, \tau_h}(c_i) | \\ &= 2A = 2A(s, L, n) \delta^s = 2k\varepsilon > 2\varepsilon. \end{aligned}$$

Hence

$$H_\varepsilon(F) \geq \log 2^h = \left(\frac{\rho}{\delta}\right)^n = \left(\frac{A(s, L, n)}{k}\right)^{\frac{n}{s}} \rho^n \left(\frac{1}{\varepsilon}\right)^{\frac{n}{s}}$$

Q.E.D.

LEMMA 2.2.3. *There exists a constant $B > 0$ such that for sufficiently small $\varepsilon > 0$*

$$H_\varepsilon(F) \leq B\rho^n \left(\frac{1}{\varepsilon}\right)^{\frac{n}{s}}$$

Proof. Let us choose some $\delta > 0$ such that the ratio ρ/δ is an integer. In the cube \mathcal{J} consider the uniform lattice with step δ , consisting of the points d_i ($i = 1, 2, \dots, h; h = \left(\frac{\rho}{\delta} + 1\right)^n$).

We shall assume the corners of the lattice to be numbered so that the point d_1 coincides with the origin of co-ordinates, and for any i

$$r(d_{i-1}, d) = \delta.$$

We now choose some function $f(x)$ of the family F and we shall show a method of constructing a table for this function the volume of which is less than $B\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s}$.

Let h_p denote the number of different kinds of partial derivative (of all orders up to and including the p -th) of a function of n variables. It is not difficult to verify that $h_p \leq (p+1)^n$. Let $\{\tau_1^{j,k}\}$ ($\tau_1^{j,k} = 0, 1$) be the coefficients of the binary representation of the numbers

$$\frac{\partial^{k_1+k_2+\dots+k_n} f(d_1)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \quad (k_1 + k_2 + \dots + k_n) \leq p$$

written in some order (k is the order of the derivative, $j = 1, 2, \dots, h_1^k$). Then the numbers

$$\left\{ \frac{\partial^{k_1+k_2+\dots+k_n} f(d_1)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \right\} \quad (k_1 + k_2 + \dots + k_n = k)$$

are represented in the table to an accuracy of δ^{s-k} , i.e.

$$h_1^k \leq \left(\left[\log \frac{c}{\delta^{s-k}} \right] + 1 \right) (k+1)^n$$

binary digits $\tau_1^{j,k}$ ($j=1, 2, \dots, h_1^k$) are sufficient to represent them in binary. Thus, to represent all partial derivatives of $f(x)$ at the point $x = d_1$ in binary we need

$$h_1 = \sum_{k=0}^p h_1^k \leq (p+1)^{n+1} \left(1 + \log \frac{c}{\delta^s} \right)$$

binary digits

$$\tau_1^{j,k} \quad (j=1, 2, \dots, h_1^k, \quad k=0, 1, 2, \dots, p).$$

Let us assume now that we have found a method for selecting the digits $\{\tau_1^{j,k}\}$ ($i=1, 2, \dots, q-1$) together with a rule for calculating from these digits the values of the numbers

$$\left\{ \frac{\partial^{k_1+k_2+\dots+k_n} f(d_i)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \right\} \quad (k_1 + k_2 + \dots + k_n = k)$$

($i=1, 2, \dots, q-1$) to an accuracy of δ^{s-k} ($k=0, 1, \dots, p$). We examine the subsequent procedure for constructing the table for $f(x)$. Let $g_k(x)$ be one of the k -th order partial derivatives of $f(x)$. According to the induction hypothesis, the values of all partial derivatives of order $m \leq p-k$ of $g_k(x)$ at the point $x = d_{q-1}$ can be calculated to an accuracy of δ^{s-k-m} ($m=0, 1, \dots, p-k$) from that part of the table already constructed. From Lagrange's formula, the value of $g_k(d_q)$ is found sufficiently accurately from the approximate values of the derivatives of $g(x)$ at d_{q-1} . Therefore, to represent the numbers $g_k(d_q)$ to an accuracy of δ^{s-k} we need only a small number of binary digits. Since $r(d_{q-1}, d_q) = \delta$ all the corresponding coordinates (except one) of the points d_{q-1}, d_q are equal. For definiteness, we shall suppose that

$$x_1(d_q) = x_1(d_{q-1}) + \delta \quad \text{and} \quad x_i(d_q) = x_i(d_{q-1})$$

for $i = 2, 3, \dots, n$. Then

$$g_k(d_q) = \sum_{m=0}^{p-m-1} \frac{\partial^m g_k(d_{q-1})}{\partial x_1^m} \cdot \frac{\delta^m}{m!}$$

$$\begin{aligned}
 & + \frac{1}{(p-1)!} \frac{\partial^{p-k} g_k (d_{q-1} + \theta \delta)}{\partial x_1^{p-k}} \delta^{p-k} \\
 & = \sum_{m=0}^{p-k} \frac{\partial^m g_k (d_{q-1})}{\partial x_1^m} \cdot \frac{\delta^m}{m!} + \frac{L}{(p-1)!} \theta \delta^{s-k},
 \end{aligned}$$

where $0 \leq \theta \leq 1$. But since $\frac{\partial^m g_k (d_{q-1})}{\partial x_1^m}$ is given by the table only to an accuracy of δ^{s-k-m} ($m=0, 1, \dots, p-k$) $g_k (d_q)$ is determined by the constructed part of the table only to an accuracy of

$$\sum_{m=0}^{p-1} \delta^{s-k-m} \frac{\delta^m}{m!} + \frac{L \delta^{s-k}}{(p-k)!} = \delta^{s-k} \left(\sum_{m=0}^{p-k} \frac{1}{m!} + \frac{L}{(p-k)!} \right) \leq e(L+1)^{s-k}$$

Therefore, in order to represent the value of $g_k (d_q)$ in the table to an accuracy of δ^{s-k} , it is sufficient to put another $h_q^{j,k} = [\log ((L+1) e)] + 1$ binary digits in the table. Hence, to determine the values of all k th order partial derivatives of $f(x)$ it is sufficient to add $h_q^k \leq (k+1)^n h_q^{j,k}$ binary digits to the table ($k=0, 1, \dots, p$). Thus, the approximate representation of the values of all partial derivatives of the functions $f(x)$ at the point will use only

$$h_q = \sum_{k=0}^p h_q^k \leq (p+1)^{n+1} (1 + \log [e(L+1)])$$

binary digits.

The volume of the table T which we have constructed is equal to

$$\begin{aligned}
 P(T) & = \sum_{q=1}^k h_q \leq (p+1)^{n+1} \left(1 + \log \frac{c}{\delta^s} \right) \\
 & + (h-1)(p+1)^{n+1} (1 + \log [e(L+1)]).
 \end{aligned}$$

We shall now describe the rule we use to enable us to compute the value of $f(x)$ at any point of the cube \mathcal{J} from the parameters of the table. To do this, we divide the cube \mathcal{J} in some way into sets ω_q ($\omega_q \ni d_q$) the diameter of each set not exceeding $\delta \sqrt{n}$, and such that $\sum_{q=1}^h \omega_q = \mathcal{J}$. The approximate value of the function $f(x)$ is calculated using the parameters $\tau_q^{j,k}$ of T in the following way.

Let $x \in \omega_q$. Then, for the approximate value of $f(x)$ we take

$$f^*(x) = \sum_{k_1+k_2+\dots+k_n \leq p} a_{k_1, k_2, \dots, k_n} \prod_{i=1}^n \frac{(x_i - x_i(d_q))^{k_i}}{k_i!}$$

where a_{k_1, k_2, \dots, k_n} is the approximate value (to an accuracy of δ^{s-k} , $k = \sum_{i=1}^n k_i$) of partial derivative $\frac{\partial^{k_1+k_2+\dots+k_n} f(d_q)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}$. Since $f(x) \in F$

$$\|f(x) - f^*(x)\| \leq \delta^s ((p+1)^m + L + 1) = B(s, L, n) \delta^s = \varepsilon'.$$

Therefore,

$$H_{\varepsilon'}(F) \leq (p+1)^{n+1} \left(1 + \log \frac{c}{\delta^s}\right) + (h-1)(p+1)^{n+1} (1 + \log (e(L+1))).$$

We now define δ in the form

$$\delta = \left(\frac{k\varepsilon}{B(s, L, n)}\right)^{1/s}$$

We choose $k < 1$ so that the ratio ρ/δ is an integer. Then

$$H_\varepsilon(F) \leq H_{\varepsilon'}(F) \leq (p+1)^{n+1} \left(1 + \log \frac{c}{\delta^s}\right) + (h-1)(p+1)^{n+1} (1 + \log (e(L+1))),$$

i.e. for sufficiently small ε $H_\varepsilon(F) \geq B\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s}$, where $B > 0$ is a constant which can be taken to depend on s, L, n only.

Q.E.D.

Proof of the Theorem 2.2.1. First let $L = 1$. Then from lemmas 2.2.2. and 2.2.3 we have

$$A\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s} \leq H_\varepsilon(F) \leq B\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s}$$

where A and B are positive constant, depending only on s and n , since in this case $L = 1$. But since

$$H_{\frac{\varepsilon}{L}}(F_{s,1,C}) = H_\varepsilon(F)$$

for sufficiently small ε

$$A(s, n) \rho^n \left(\frac{L}{\varepsilon}\right)^{n/s} \leq H_\varepsilon(F) \leq B(s, n) \rho^n \left(\frac{L}{\varepsilon}\right)^{n/s}$$

Q.E.D.