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$$H_{\varepsilon}(F_{\rho_1, \rho_2, \dots, \rho_n}^c) = \frac{1}{(n+1)!} \prod_{k=1}^n \frac{1}{\log \rho_k} \left( \log \frac{c}{\varepsilon} \right)^{n+1} + O \left[ \left( \log \frac{c}{\varepsilon} \right)^n \log \log \frac{c}{\varepsilon} \right].$$

3. Let  $F_{s,c}^n$  be the class of real valued functions on the cube  $\{-1 \leq x_k \leq 1\}$  ( $k=1, \dots, n$ ), bounded in modulus on that cube by the constant  $s_k$  and such that their analytic extensions are entire functions of order  $s_k$  with respect to  $z_k = x_k + iy_k$  ( $k=1, \dots, n$ ). Then

$$\begin{aligned} H_{\varepsilon}(F_{s,c}^n) &= \frac{1}{(n+1)!} \prod_{k=1}^n s_k \left( \log \frac{c}{\varepsilon} \right)^{n+1} \left( \log \log \frac{c}{\varepsilon} \right)^{-n} + \\ &= O \left[ \left( \log \frac{c}{\varepsilon} \right)^{n+1} \left( \log \log \frac{c}{\varepsilon} \right)^{-n-1} \right]. \end{aligned}$$

These estimates and other results connected with estimates of entropy and applications are to be found for example in [49]-[53].

## § 2. The entropy of the space of smooth functions

Here we give an estimate of the entropy of the class of  $S$  times differentiable functions of  $n$  variables. The lower estimate was obtained in [4], the upper one—in [23].

We fix integers  $n \geq 1$  and  $p \geq 0$  and numbers  $0 \leq \alpha \leq 1$ ,  $L > 0$ ,  $C > 0$ ,  $\rho > 0$ . We will denote by  $\mathcal{J}$  the cube  $0 \leq x_i \leq \rho$  ( $i=1, \dots, n$ ) and by  $F = F_{S,L,c}^{\rho,n}$  ( $S=p+\alpha$ ) the set of all real valued functions defined on  $\mathcal{J}$  such that their partial derivatives of order  $p$  satisfy the condition  $\text{Lip } \alpha$  with the constant  $L$  and

$$\left| \frac{\partial^{k_1+\dots+k_n} f(0)}{\partial^{k_1} x_1 \dots \partial^{k_n} x_n} \right| \leq c \left( \sum_{i=1}^n k_i \leq p \right)$$

We say that the function  $g(x)$  satisfies the condition  $\text{Lip } \alpha$  with the constant  $L$  if for any  $x'$  and  $x''$

$$|g(x') - g(x'')| \leq L(r(x', x''))^{\alpha},$$

where  $r(x', x'')$  is the distance between  $x'$  and  $x''$ .

**THEOREM 2.2.1.** *If  $\varepsilon > 0$  is sufficiently small then*

$$A\rho^n \left( \frac{L}{\varepsilon} \right)^{n/s} \leq H_{\varepsilon}(F) \leq B\rho^n \left( \frac{L}{\varepsilon} \right)^{n/s},$$

where  $A$  and  $B$  are positive constants depending only on  $s$  and  $n$ .

We choose  $\delta > 0$  such that the number  $\rho/\delta$  is an integer. We divide the cube  $\mathcal{J}$  into  $\left(\frac{\rho}{\delta}\right)^n$  cubes  $P_i$  ( $i = 1, 2, \dots, \left(\frac{\rho}{\delta}\right)^n$ ) by hyperplanes, parallel to its  $(n-1)$ -dimensional edges. Each of the cubes  $P_i$  has side of length  $\delta$ , and the edges of these cubes are parallel to those on  $\mathcal{J}$ . Let  $C_i$  denote the centre of the cube  $P_i$  and  $S_i$  the  $n$ -dimensional closed sphere (inscribed in  $P_i$ ) of radius  $\delta/2$  and centre at the point  $C_i$ . Put

$$\varphi_i(x) = \varphi_i(x_1, x_2, \dots, x_n) = \begin{cases} 0, & \text{if } x \in \mathcal{J} - S_i \\ A \left( 1 + \cos \left( \frac{2\pi}{\delta} r(C_i, x) \right) \right)^p & \text{if } x \in S_i, \end{cases}$$

where  $r(C_i, x)$  is the distance from the point  $x$  to the centre  $C_i$  of the sphere  $S_i$ . Put, further,

$$\varphi_{\eta_1, \eta_2, \dots, \eta_h}(x) = \sum_{i=1}^h \eta_i \varphi_i(x) \\ \left( \eta_i = \pm 1; i = 1, 2, \dots, h; h = \left(\frac{\rho}{\delta}\right)^n \right).$$

LEMMA 2.2.1. *We can find a positive number  $A(s, L, n)$ , such that when  $A = A(s, L, n) \delta^s$  and given any set of numbers  $\eta_i$  ( $i = 1, 2, \dots, h$ )-the corresponding function  $\varphi_{\eta_1, \eta_2, \dots, \eta_h}(x)$  belongs to  $F$ .*

*Proof.* By differentiating  $\varphi_i(x)$  it is not difficult to see that inside the sphere  $S_i$  its partial derivatives of all orders exist. And the modulus of any partial derivative of order  $k$  is bounded inside  $S_i$  by  $AB(s, k, n) \delta^{-k}$ , where  $B(s, k, n)$  is some constant, depending only on  $s, k, n$ . In particular, any derivative of the function  $\varphi_i(x)$  of order  $p+1$  is bounded in the sphere  $S_i$  by the constant

$$AB(s, p+1, n) \delta^{-p-1} = \frac{A(s, L, n) B(s, p+1, n)}{\delta^{1-\alpha}}.$$

Let  $g(x)$  be any  $p$ -th order partial derivative of the functions  $\varphi_i(x)$ . We take two points  $a$  and  $b$  belonging to the sphere  $S_i$ . Then  $g(b) - g(a) = r(a, b) \frac{\partial g(c)}{\partial r}$ , where  $\frac{\partial g(c)}{\partial r}$  is the derivative of  $g(x)$  along the direction  $(a, b)$ , taken at some point  $c$  of  $[a, b]$ . Since any  $p+1$ -th order partial derivative of  $\varphi_i(x)$  is bounded inside the sphere by the constant

$$\frac{A(s, L, n) B(s, p+1, n)}{\delta^{1-\alpha}}, \text{ we have } \left| \frac{\partial g(c)}{\partial r} \right| \leq n \frac{A(s, L, n) B(s, p+1, n)}{\delta^{1-\alpha}}$$

And then

$$\begin{aligned} |g(b) - g(a)| &\leq \left| \rho \frac{\partial g(c)}{\partial r} \right| \leq \rho n \frac{A(s, L, n) B(s, p+1, n)}{\delta^{1-\alpha}} \\ &\leq \rho^\alpha n A(s, L, n) B(s, p+1, n). \end{aligned}$$

Put

$$A(s, L, n) = \frac{L}{2n B(s, p+1, n)}.$$

Then

$$|g(b) - g(a)| \leq \frac{1}{2} L \rho^\alpha.$$

Now let  $\Psi(x)$  be any of the  $p$ -th partial derivatives of the function  $\varphi_{\eta_1, \eta_2, \dots, \eta_h}(x)$ . We choose two points  $x'$  and  $x''$  of  $\mathcal{J}$  ( $x' \in S_i, x'' \in S_j$ ) and let  $g_1(x)$  and  $g_2(x)$  be the partial derivatives of the same kind as  $\Psi(x)$  of the functions  $\varphi_i(x)$  and  $\varphi_j(x)$  (respectively). It is easy to verify that  $g_1(x)$  and  $g_2(x)$  are continuous on  $\mathcal{J}$  and identically equal to zero on the sets  $\mathcal{J} - S_i$  and  $\mathcal{J} - S_j$  (respectively). We select some point  $x_0$  belonging to the boundary of the sphere  $S_i$  and lying on the segment  $[x', x'']$ .

Then

$$\begin{aligned} |\psi(x'') - \psi(x')| &\leq |g_1(x'') - g_1(x')| + |g_2(x'') - g_2(x')| \\ &\leq |g_1(x') - g_1(x_0)| + |g_2(x'') - g_2(x_0)| \leq |g(b) - g(a)| \\ &\leq \frac{1}{2} L (r(x', x_0))^\alpha + \frac{1}{2} L (r(x'', x_0))^\alpha \leq L (r(x', x''))^\alpha. \end{aligned}$$

If one of the points  $x', x''$  (or both) belongs to the set  $\mathcal{J} - \sum_{i=1}^h S_i$ , then we can prove similarly that

$$|\varphi(x'') - \varphi(x')| \leq L (r(x', x''))^\alpha.$$

Q.E.D.

LEMMA 2.2.2. *There exists a positive constant  $A$ , depending only on  $s, L, n$  such that for sufficiently small  $\varepsilon$*

$$H_\varepsilon(F) \geq A \rho^n \left( \frac{1}{\varepsilon} \right)^{n/s}.$$

*Proof.* We choose some positive number  $k > 1$  such that when  $\delta = \left( \frac{k\varepsilon}{A(s, L, n)} \right)^{1/s}$  is an integer.

We choose two different functions of the type  $\varphi_{\eta_1, \dots, \eta_h}(x)$  and  $\varphi_{\tau_1, \tau_2, \dots, \tau_h}(x)$ ,  $A = A(s, L, n) \delta^s$  and  $A(s, L, n)$  is taken so small that both functions belong to the family  $F$ . Since the functions we have chosen are assumed to be different, for some  $i$   $\tau_i \neq \eta_i$ . And therefore

$$\begin{aligned} & |\varphi_{\eta_1, \eta_2, \dots, \eta_h}(c_i) - \varphi_{\tau_1, \tau_2, \dots, \tau_h}(c_i)| \\ &= 2A = 2A(s, L, n) \delta^s = 2k\varepsilon > 2\varepsilon. \end{aligned}$$

Hence

$$H_\varepsilon(F) \geq \log 2^h = \left(\frac{\rho}{\delta}\right)^n = \left(\frac{A(s, L, n)}{k}\right)^{\frac{n}{s}} \rho^n \left(\frac{1}{\varepsilon}\right)^{\frac{n}{s}}$$

Q.E.D.

LEMMA 2.2.3. *There exists a constant  $B > 0$  such that for sufficiently small  $\varepsilon > 0$*

$$H_\varepsilon(F) \leq B\rho^n \left(\frac{1}{\varepsilon}\right)^{\frac{n}{s}}$$

*Proof.* Let us choose some  $\delta > 0$  such that the ratio  $\rho/\delta$  is an integer. In the cube  $\mathcal{J}$  consider the uniform lattice with step  $\delta$ , consisting of the points  $d_i$  ( $i = 1, 2, \dots, h$ ;  $h = \left(\frac{\rho}{\delta} + 1\right)^n$ ).

We shall assume the corners of the lattice to be numbered so that the point  $d_1$  coincides with the origin of co-ordinates, and for any  $i$

$$r(d_{i-1}, d) = \delta.$$

We now choose some function  $f(x)$  of the family  $F$  and we shall show a method of constructing a table for this function the volume of which is less than  $B\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s}$ .

Let  $h_p$  denote the number of different kinds of partial derivative (of all orders up to and including the  $p$ -th) of a function of  $n$  variables. It is not difficult to verify that  $h_p \leq (p+1)^n$ . Let  $\{\tau_1^{j,k}\}$  ( $\tau_1^{j,k} = 0, 1$ ) be the coefficients of the binary representation of the numbers

$$\frac{\partial^{k_1+k_2+\dots+k_n} f(d_1)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \quad (k_1 + k_2 + \dots + k_n \leq p)$$

written in some order ( $k$  is the order of the derivative,  $j = 1, 2, \dots, h_1^k$ ). Then the numbers

$$\left\{ \frac{\partial^{k_1+k_2+\dots+k_n} f(d_1)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \right\} \quad (k_1 + k_2 + \dots + k_n = k)$$

are represented in the table to an accuracy of  $\delta^{s-k}$ , i.e.

$$h_1^k \leq \left( \left[ \log \frac{c}{\delta^{s-k}} \right] + 1 \right) (k+1)^n$$

binary digits  $\tau_1^{s,k}$  ( $j=1, 2, \dots, h_1^k$ ) are sufficient to represent them in binary. Thus, to represent all partial derivatives of  $f(x)$  at the point  $x = d_1$  in binary we need

$$h_1 = \sum_{k=0}^p h_1^k \leq (p+1)^{n+1} \left( 1 + \log \frac{c}{\delta^s} \right)$$

binary digits

$$\tau_1^{j,k} \quad (j=1, 2, \dots, h_1^k, \quad k=0, 1, 2, \dots, p).$$

Let us assume now that we have found a method for selecting the digits  $\{\tau_1^{j,k}\}$  ( $i=1, 2, \dots, q-1$ ) together with a rule for calculating from these digits the values of the numbers

$$\left\{ \frac{\partial^{k_1+k_2+\dots+k_n} f(d_i)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}} \right\} \quad (k_1 + k_2 + \dots + k_n = k)$$

( $i=1, 2, \dots, q-1$ ) to an accuracy of  $\delta^{s-k}$  ( $k=0, 1, \dots, p$ ). We examine the subsequent procedure for constructing the table for  $f(x)$ . Let  $g_k(x)$  be one of the  $k$ -th order partial derivatives of  $f(x)$ . According to the induction hypothesis, the values of all partial derivatives of order  $m \leq p-k$  of  $g_k(x)$  at the point  $x = d_{q-1}$  can be calculated to an accuracy of  $\delta^{s-k-m}$  ( $m=0, 1, \dots, p-k$ ) from that part of the table already constructed. From Lagrange's formula, the value of  $g_k(d_q)$  is found sufficiently accurately from the approximate values of the derivatives of  $g(x)$  at  $d_{q-1}$ . Therefore, to represent the numbers  $g_k(d_q)$  to an accuracy of  $\delta^{s-k}$  we need only a small number of binary digits. Since  $r(d_{q-1}, d_q) = \delta$  all the corresponding coordinates (except one) of the points  $d_{q-1}, d_q$  are equal. For definiteness, we shall suppose that

$$x_1(d_q) = x_1(d_{q-1}) + \delta \quad \text{and} \quad x_i(d_q) = x_i(d_{q-1})$$

for  $i = 2, 3, \dots, n$ . Then

$$g_k(d_q) = \sum_{m=0}^{p-k-1} \frac{\partial^m g_k(d_{q-1})}{\partial x_1^m} \cdot \frac{\delta^m}{m!}$$

$$+ \frac{1}{(p-1)!} \frac{\partial^{p-k} g_k(d_{q-1} + \theta \delta)}{\partial x_1^{p-k}} \delta^{p-k}$$

$$= \sum_{m=0}^{p-k} \frac{\partial^m g_k(d_{q-1})}{\partial x_1^m} \cdot \frac{\delta^m}{m!} + \frac{L}{(p-1)!} \theta \delta^{s-k},$$

where  $0 \leq \theta \leq 1$ . But since  $\frac{\partial^m g_k(d_{q-1})}{\partial x_1^m}$  is given by the table only to an accuracy of  $\delta^{s-k-m}$  ( $m=0, 1, \dots, p-k$ )  $g_k(d_q)$  is determined by the constructed part of the table only to an accuracy of

$$\sum_{m=0}^{p-1} \delta^{s-k-m} \frac{\delta^m}{m!} + \frac{L \delta^{s-k}}{(p-k)!} = \delta^{s-k} \left( \sum_{m=0}^{p-k} \frac{1}{m!} + \frac{L}{(p-k)!} \right) \leq e(L+1)^{s-k}$$

Therefore, in order to represent the value of  $g_k(d_q)$  in the table to an accuracy of  $\delta^{s-k}$ , it is sufficient to put another  $h_q^{j,k} = [\log((L+1)e)] + 1$  binary digits in the table. Hence, to determine the values of all  $k$  th order partial derivatives of  $f(x)$  it is sufficient to add  $h_q^k \leq (k+1)^n h_q^{j,k}$  binary digits to the table ( $k=0, 1, \dots, p$ ). Thus, the approximate representation of the values of all partial derivatives of the functions  $f(x)$  at the point will use only

$$h_q = \sum_{k=0}^p h_q^k \leq (p+1)^{n+1} (1 + \log [e(L+1)])$$

binary digits.

The volume of the table  $T$  which we have constructed is equal to

$$P(T) = \sum_{q=1}^k h_q \leq (p+1)^{n+1} \left( 1 + \log \frac{c}{\delta^s} \right)$$

$$+ (h-1)(p+1)^{n+1} (1 + \log [e(L+1)]).$$

We shall now describe the rule we use to enable us to compute the value of  $f(x)$  at any point of the cube  $\mathcal{J}$  from the parameters of the table. To do this, we divide the cube  $\mathcal{J}$  in some way into sets  $\omega_q$  ( $\omega_q \ni d_q$ ) the diameter of each set not exceeding  $\delta \sqrt{n}$ , and such that  $\sum_{q=1}^h \omega_q = \mathcal{J}$ . The approximate value of the function  $f(x)$  is calculated using the parameters  $\tau_q^{j,k}$  of  $T$  in the following way.

Let  $x \in \omega_q$ . Then, for the approximate value of  $f(x)$  we take

$$f^*(x) = \sum_{k_1+k_2+\dots+k_n \leq p} a_{k_1, k_2, \dots, k_n} \prod_{i=1}^n \frac{(x_i - x_i(d_q))^{k_i}}{k_i!}$$

where  $a_{k_1, k_2, \dots, k_n}$  is the approximate value (to an accuracy of  $\delta^{s-k}$ ,  $k = \sum_{i=1}^n k_i$ ) of partial derivative  $\frac{\partial^{k_1+k_2+\dots+k_n} f(d_q)}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}$ . Since  $f(x) \in F$

$$\|f(x) - f^*(x)\| \leq \delta^s ((p+1)^m + L + 1) = B(s, L, n) \delta^s = \varepsilon'.$$

Therefore,

$$H_{\varepsilon'}(F) \leq (p+1)^{n+1} \left(1 + \log \frac{c}{\delta^s}\right) + (h-1)(p+1)^{n+1} (1 + \log(e(L+1))).$$

We now define  $\delta$  in the form

$$\delta = \left(\frac{k\varepsilon}{B(s, L, n)}\right)^{1/s}$$

We choose  $k < 1$  so that the ratio  $\rho/\delta$  is an integer. Then

$$H_{\varepsilon}(F) \leq H_{\varepsilon'}(F) \leq (p+1)^{n+1} \left(1 + \log \frac{c}{\delta^s}\right) + (h-1)(p+1)^{n+1} (1 + \log(e(L+1))),$$

i.e. for sufficiently small  $\varepsilon$   $H_{\varepsilon}(F) \geq B\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s}$ , where  $B > 0$  is a constant which can be taken to depend on  $s, L, n$  only.

Q.E.D.

*Proof of the Theorem 2.2.1.* First let  $L = 1$ . Then from lemmas 2.2.2. and 2.2.3 we have

$$A\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s} \leq H_{\varepsilon}(F) \leq B\rho^n \left(\frac{1}{\varepsilon}\right)^{n/s}$$

where  $A$  and  $B$  are positive constant, depending only on  $s$  and  $n$ , since in this case  $L = 1$ . But since

$$H_{\frac{\varepsilon}{L}}(F_{s,1,C}) = H_{\varepsilon}(F)$$

for sufficiently small  $\varepsilon$

$$A(s, n) \rho^n \left(\frac{L}{\varepsilon}\right)^{n/s} \leq H_{\varepsilon}(F) \leq B(s, n) \rho^n \left(\frac{L}{\varepsilon}\right)^{n/s}$$

Q.E.D.