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SUPERPOSITIONS AND RELATED TOPICS  
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the same time, from the inequalities mentioned above it follows that an increase in  $k$  leads to an insignificant improvement in the accuracy of the approximation. Hence, in a certain sense, a more economical approximation of functions is by means of expressions of the given form with  $k = 1$ , that is, by fractions of the form

$$\frac{\sum_{i=0}^p a_i f_i(x)}{\sum_{j=0}^p b_j g_j(x)}$$

The same inequalities with  $k = 1$  show that there are no methods of approximating functions by fractions of the given form essentially better than the standard methods of approximating functions by algebraic (or trigonometric) polynomials.

#### § 4. *Superpositions of continuous functions*

Kolmogorov's theorem on the possibility of representing continuous functions of  $n$  variables as superpositions of continuous functions of three variables was highly unexpected (see [7]).

In this paper Kolmogorov proves that on the  $n$ -dimensional cube  $\mathcal{J}^n$  we can construct continuous functions  $\varphi_i(x)$  ( $i = 1, 2, \dots, n+1$ ) such that any continuous function  $f(x)$ , defined on the cube  $\mathcal{J}^n$ , can be represented in the form

$$f(x) = \sum_{i=1}^{n+1} f_i(d_i(x)),$$

where  $d_i(x)$  is a continuous mapping of  $\mathcal{J}^n$  onto the one-dimensional tree <sup>1)</sup>  $D$  if the components of the level sets of the functions  $\varphi_i(x)$ , and  $f_i(d_i)$  is a continuous function on the tree  $D_i$ . Since the trees  $\{D_i\}$  can be embedded homeomorphically in the plane (see [30]), the functions  $\{f_i(d_i(x))\}$  can be thought of as superpositions

$$\{f_i(u_i(x_1, x_2, \dots, x_n), v_i(x_1, x_2, \dots, x_n))\}$$

<sup>1)</sup> Kronrod [29] has shown that the components of all possible level sets of any continuous function defined on  $\mathcal{J}^n$  in a certain natural topology, form a tree, that is, a one-dimensional locally connected continuum, not containing homeomorphic images of circles. Kronrod calls this the "one-dimensional tree of the function".

where  $\{f_i(u_i, v_i)\}$  are continuous functions of two variables, and  $\{u_i(x)\}$  and  $\{v_i(x)\}$  are fixed continuous functions of  $n$  variables. Kolmogorov derived from this the result that for  $n \geq 4$  any continuous function of  $n$  variables can be represented by the following superposition of continuous functions of not more than  $n - 1$  variables:

$$\sum_{i=1}^n f_i(u_i(x_1, x_2, \dots, x_{n-1}), v_i(x_1, x_2, \dots, x_{n-1}), x_n).$$

Arnol'd [8], [22] showed that, firstly, in Kolmogorov's construction [7] we can manage with functions  $\{\varphi_i(x)\}$  whose one-dimensional trees  $\{D_i\}$  have index at each branch point equal to 3, and, secondly, for any compact set  $F$  of functions defined on such a tree  $D$ , the given tree can be so placed in three-dimensional  $u, v, w$ -space that any continuous function  $f(d) = f(u, v, w) \in F$  can be represented as the sum of functions of the coordinates,  $f(u, v, w) = \varphi(u) + \psi(v) + \kappa(w)$ . Hence it follows that any continuous function  $f(x, y, z)$  of three variables can be represented as a superposition of the form  $f(x, y, z) = \sum_{i=1}^9 f_i(\varphi_i(x, y), z)$ , where all the functions are continuous, and the functions  $\{\varphi_i(x, y)\}$  can be regarded as fixed, when  $f(x, y, z)$  is taken from a compact set. Thus, Arnol'd had the last word in refuting Hilbert's conjecture. At the same time Kolmogorov [9] obtained, in a certain sense, the definitive result in this direction.

Each continuous function of  $n$  variables, given on the unit cube in  $n$ -dimensional space, is representable as a superposition of the form

$$f(x_1, x_2, \dots, x_n) = \sum_{q=1}^{2n+1} g_q\left(\sum_{p=1}^n \varphi_{p,q}(x_p)\right), \quad (\text{I})$$

where all the functions are continuous, and moreover the functions  $\{\varphi_{p,q}(x_p)\}$  are standard and monotonic.

In particular, each continuous function of two variables is representable in the form

$$f(x, y) = \sum_{i=1}^5 f_i(a_i(x) + \beta_i(y)). \quad (\text{II})$$

Kolmogorov's theorem can be supplemented by the following result of Bari, which was obtained in connection with problems of Fourier series: any continuous function of one variable  $f(t)$  can be represented in the form  $f(t) = f_1(\varphi_1(t)) + f_2(\varphi_2(t)) + f_3(\varphi_3(t))$ , where all the functions  $\{f_i\}$  and  $\{\varphi_i\}$  are absolutely continuous [32].

From the theorems of Kolmogorov and Bari it follows that each continuous function of  $n$  variables can be represented as a superposition of absolutely continuous monotonic functions of one variable and the operation of addition.

A detailed account of Kolmogorov's theorem is to be found in the surveys [9], [33]-[36]. The proof presented by Kahane is of special interest [36]. He does not attempt to construct the functions  $\{ \varphi_{p,q} \}$  (as the proof of Kolmogorov does) but instead he shows by means of Baire's theorem, that most selections of increasing functions  $\{ \varphi_{pq} \}$  will do. This approach also lead to other interesting results.

Fridman [37] showed that the inner functions  $\{ \varphi_{pq} \}$  can be chosen from the class Lip 1. Kahane noticed that this follows directly from Kolmogorov's theorem. For any finite collection of continuous and monotone functions  $\{ f_k(x) \}$  on the segment  $[0, 1]$  there exists a homeomorphism  $x = \varphi(s)$  of the segment  $[0, 1]$  onto itself such that the functions  $\{ g_k(s) = f_k(\varphi(s)) \}$  belong to the class Lip 1. The homeomorphism is taken as

$$s = \varphi^{-1}(x) = \varepsilon \left( x + \sum_k |f_k(x) - f_k(0)| \right).$$

The constant  $\varepsilon$  is chosen to satisfy the condition  $\varphi^{-1}(1) = 1$ . By means of such homeomorphisms all inner functions in Kolmogorov's formula can be turned into functions satisfying the condition Lip 1.

There are some other improvements of Kolmogorov's theorem: Doss [38], Bassalygo [39], Lorentz [34], Sprecher [40] (see chapter 3, § 1). There are also many results concerning special types of superpositions (see [21], [33], [41]-[44]).

## § 5. *Linear superpositions*

We return again to superpositions of smooth functions.

One of the most interesting current problems on the subject of superpositions is the following: does there exist an analytic function of two variables that cannot be represented as a finite superposition of continuously differentiable (smooth) functions of one variable and the operation of addition?

Linear superpositions arise as a result of the following argument. Suppose that a function of two variables  $f(x, y)$  is an  $s$ -fold superposition of certain smooth functions of one variable  $\{ f_i(t) \}$  and the operation of addition. We vary this superposition, that is, we consider a superposition