Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 23 (1977)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS

Autor: Vitushkin, A. G.

Kapitel: §4. Superpositions of continuons functions

DOI: https://doi.org/10.5169/seals-48931

the same time, from the inequalities mentioned above it follows that an increase in k leads to an insignificant improvement in the accuracy of the approximation. Hence, in a certain sense, a more economical approximation of functions is by means of expressions of the given form with $k = 1$, that is, by fractions of the form

$$\frac{\sum_{i=0}^{p} a_i f_i(x)}{\sum_{j=0}^{p} b_j g_j(x)}$$

The same inequalities with $k = 1$ show that there are no methods of approximating functions by fractions of the given form essentially better than the standard methods of approximating functions by algebraic (or trigonometric) polynomials.

§ 4. *Superpositions of continuous functions*

Kolmogorov's theorem on the possibility of representing continuous functions of n variables as superpositions of continuous functions of three variables was highly unexpected (see [7]).

In this paper Kolmogorov proves that on the n-dimensional cube $\mathscr{I}^n$ we can construct continuous functions $\varphi_i(x)$ $(i = 1, 2, \ldots, n+1)$ such that any continuous function $f(x)$, defined on the cube $\mathscr{I}^n$, can be represented in the form

$$f(x) = \sum_{i=1}^{n+1} f_i(d_i(x)),$$

where $d_i(x)$ is a continuous mapping of $\mathscr{I}^n$ onto the one-dimensional tree [1]) D if the components of the level sets of the functions $\varphi_i(x)$, and $f_i(d_i)$ is a continuous function on the tree D_i. Since the trees $\{D_i\}$ can be embedded homeomorphically in the plane (see [30]), the functions $\{f_i(d_i(x))\}$ can be thought of as superpositions

$$\{f_i(u_i(x_1, x_2, \ldots, x_n), v_i(x_1, x_2, \ldots, x_n))\}$$

[1]) Kronrod [29] has shown that the components of all possible level sets of any continuous function defined on $\mathscr{I}^n$ in a certain natural topology, form a tree, that is, a one-dimensional locally connected continuum, not containing homeomorphic images of circles. Kronrod calls this the "one-dimensional tree of the function".

where $\{f_i(u_i, v_i)\}$ are continuous functions of two variables, and $\{u_i(x)\}$ and $\{v_i(x)\}$ are fixed continuous functions of n variables. Kolmogorov derived from this the result that for $n \geqslant 4$ any continuous function of n variables can be represented by the following superposition of continuous functions of not more than $n-1$ variables:

$$\sum_{i=1}^{n} f_i\big(u_i(x_1, x_2, \dots, x_{n-1}),\ v_i(x_1, x_2, \dots, x_{n-1}),\ x_n\big).$$

Arnol'd [8], [22] showed that, firstly, in Kolmogorov's construction [7] we can manage with functions $\{\varphi_i(x)\}$ whose one-dimensional trees $\{D_i\}$ have index at each branch point equal to 3, and, secondly, for any compact set F of functions defined on such a tree D, the given tree can be so placed in three-dimensional u, v, w-space that any continuous function $f(d) = f(u, v, w) \in F$ can be represented as the sum of functions of the coordinates, $f(u, v, w) = \varphi(u) + \psi(v) + \kappa(w)$. Hence it follows that any continuous function $f(x, y, z)$ of three variables can be represented as a superposition of the form $f(x, y, z) = \sum_{i=1}^{9} f_i(\varphi_i(x, y), z)$, where all the functions are continuous, and the functions $\{\varphi_i(x, y)\}$ can be regarded as fixed, when $f(x, y, z)$ is taken from a compact set. Thus, Arnol'd had the last word in refuting Hilbert's conjecture. At the same time Kolmogorov [9] obtained, in a certain sense, the definitive result in this direction.

Each continuous function of n variables, given on the unit cube in n-dimensional space, is representable as a superposition of the form

$$f(x_1, x_2, \dots, x_n) = \sum_{q=1}^{2n+1} g_q\Big(\sum_{p=1}^{n} \varphi_{p,q}(x_p)\Big), \tag{I}$$

where all the functions are continuous, and moreover the functions $\{\varphi_{p,q}(x_p)\}$ are standard and monotonic.

In particular, each continuous function of two variables is representable in the form

$$f(x, y) = \sum_{i=1}^{5} f_i(\alpha_i(x) + \beta_i(y)). \tag{II}$$

Kolmogorov's theorem can be supplemented by the following result of Bari, which was obtained in connection with problems of Fourier series: any continuous function of one variable $f(t)$ can be represented in the form $f(t) = f_1(\varphi_1(t)) + f_2(\varphi_2(t)) + f_3(\varphi_3(t))$, where all the functions $\{f_i\}$ and $\{\varphi_i\}$ are absolutely continuous [32].

From the theorems of Kolmogorov and Bari it follows that each continuous function of n variables can be represented as a superposition of absolutely continuous monotonic functions of one variable and the operation of addition.

A detailed account of Kolmogorov's theorem is to be found in the surveys [9], [33]-[36]. The proof presented by Kahane is of special interest [36]. He does not attempt to construct the functions $\{ \varphi_{p,q} \}$ (as the proof of Kolmogorov does) but instead he shows by means of Baire's theorem, that most selections of increasing functions $\{ \varphi_{pq} \}$ will do. This approach also lead to other interesting results.

Fridman [37] showed that the inner functions $\{ \varphi_{pq} \}$ can be chosen from the class Lip 1. Kahane noticed that this follows directly from Kolmogorov's theorem. For any finite collection of continuous and monotone functions $\{ f_k (x) \}$ on the segment [0, 1] there exists a homeomorphism $x = \varphi (s)$ of the segment [0, 1] onto itself such that the functions $\{ g_k (s) = f_k (\varphi (s)) \}$ belong to the class Lip 1. The homeomorphism is taken as

$$s = \varphi^{-1} (x) = \varepsilon \left(x + \sum_k |f_k (x) - f_k (0)|\right).$$

The constant ε is chosen to satisfy the condition $\varphi^{-1} (1) = 1$. By means of such homeomorphisms all inner functions in Kolmogorov's formula can be turned into functions satisfying the condition Lip 1.

There are some other improvements of Kolmogorov's theorem: Doss [38], Bassalygo [39], Lorentz [34], Sprecher [40] (see chapter 3, § 1). There are also many results concerning special types of superpositions (see [21], [33], [41]-[44]).

§ 5. *Linear superpositions*

We return again to superpositions of smooth functions.

One of the most interesting current problems on the subject of superpositions is the following: does there exist an analytic function of two variables that cannot be represented as a finite superposition of continuously differentiable (smooth) functions of one variable and the operation of addition?

Linear superpositions arise as a result of the following argument. Suppose that a function of two variables $f(x, y)$ is an s-fold superposition of certain smooth functions of one variable $\{ f_i (t) \}$ and the operation of addition. We vary this superposition, that is, we consider a superposition