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QUADRATIC FORMS IN AN ADELIC SETTING [1])

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1. INTRODUCTION

The connection between Siegel's main theorem in the analytical theory of quadratic forms and the determination of the Tamagawa number of the orthogonal group has been discussed in expository articles by Kneser [1] and Tamagawa [5]. Both of these papers, however, consider only a special case of Siegel's general theorem, namely the number of representations of a quadratic form by itself. In the present paper we consider the problem of representing a positive definite form by another positive definite form. Siegel's main theorem is derived from an adelic integral formula, Ono's "mean value theorem", which is the analogue for the adelized orthogonal group of Siegel's mean value theorem in the geometry of numbers.

2. THE MEAN VALUE FORMULA

The adelic mean value formula generalizes Siegel's mean value theorem in the geometry of numbers [3]. We first describe Siegel's theorem as reformulated by Weil [8].

Let Φ be a continuous function of $\mathbf{R}^n$ ($n \geqslant 2$) with compact support. Then

$$\Phi(g) = \sum_{X \in \mathbf{Z}^n - \{0\}} \Phi(gx)$$

defines a function on $SL_n(\mathbf{R})$, right invariant by $SL_n(\mathbf{Z})$. According to Siegel's theorem, $SL_n(\mathbf{R}) / SL_n(\mathbf{Z})$ has finite measure, Φ is integrable on this space, and

$$\frac{\int_{SL_n(\mathbf{R})/SL_n(\mathbf{Z})} \Phi(g)\, dg}{\int_{SL_n(\mathbf{R})/SL_n(\mathbf{Z})} dg} = \int_{\mathbf{R}^n} \Phi(x)\, dx\,.$$

In Siegel's original formulation, Φ is taken to be the characteristic function

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[1]) The author would like to express his appreciation to Professor T. Ono for his valuable advice.

of a compact set K. The right-hand side of the above formula reduces to the volume of K, while the left-hand side gives the mean value of

$$\mathrm{card}\,(L - \{0\} \cap K)\,,$$

as L varies over all $\mathbf{Z}$-lattices in $\mathbf{R}^n$ with volume 1.

We turn now to the adelic mean value formula. Let G be a linear algebraic group defined over $\mathbf{Q}$, and let X be an algebraic homogeneous space for G, defined over $\mathbf{Q}$. For $\xi \in X$, let $G_\xi = \{g \in G : g\xi = \xi\}$. We assume that

a) X has at least one rational point

b) for any $\xi \in X_{\mathbf{C}}$, both $G_{\mathbf{C}}$ and $(G_\xi)_{\mathbf{C}}$ have finite fundamental groups

c) for any extension field K of $\mathbf{Q}$, G_K acts transitively on X_K.

We then have the following result.

THEOREM (Ono [2]). There are canonical measures on the adele spaces G_A and X_A such that, given any continuous function Φ on X_A with compact support,

$$(A) \qquad \frac{\int_{G_A/G_{\mathbf{Q}}} \sum_{x \in X_{\mathbf{Q}}} \Phi(gx)\, dg}{\tau(G_\xi)} = \int_{X_A} \Phi(x)\, dx\,,$$

where ξ is any element of $X_{\mathbf{Q}}$, and $\tau(G_\xi)$ = the invariant measure of $(G_\xi)_A \,/\, (G_\xi)_{\mathbf{Q}}$. The analogy to the previous mean value theorem is clear in the cases when $\tau(G) = \tau(G_\xi)$.

3. FORMULATION OF SIEGEL'S THEOREM

Let S and T be square matrices with integral entries of size m and n, respectively. We assume that both are positive definite. For any matrix x, denote $S[x] = {}^t xSx$ (when defined). Let $A(S, T)$ = the number of integral $m \times n$ matrices x such that $S[x] = T$. For each positive integer q, let $A_q(S, T)$ = the number of integral $m \times n$ matrices x, mod q, such that $S[x] \equiv T \pmod q$.

A positive definite integral matrix S' is said to be in the same class as S if $S' = S[U]$, for some $U \varepsilon SL(m, \mathbf{Z})$. S' is in the same genus as S if for each q, there exists $U \varepsilon SL(m, \mathbf{Z})$ such that $S' \equiv S[U] \pmod q$. Let $S_1, \ldots, S_h$ be the representatives of the classes in genus (S). Let $E(S_i)$ = the finite group consisting of all $U \varepsilon SL(m, \mathbf{Z})$ such that $S_i[U] = S_i$, and put

$e_i = 1 / \# E(S_i)$, where $\#$ denotes cardinality. We now define the "number of representations of T by the genus of S" as

$$A(\text{genus}(S), T) = \frac{e_1 A(S_1, T) + \dots + e_h A(S_h, T)}{e_1 + \dots + e_h}.$$

Now S is a real symmetric matrix, and so we may view it as a point in $\mathbf{R}^{n_1}$, where $n_1 = n(n+1)/2$. Similarly, T is a point in $\mathbf{R}^{m_1}$. Let dt be the usual measure in $\mathbf{R}^{m_1}$, and let dx be the usual measure in the real vector space of $m \times n$ matrices. Given $\varepsilon > 0$, let B_ε denote the ε-neighborhood of T in $\mathbf{R}^{m_1}$, and let C_ε denote the set of $x \,\varepsilon\, M_{m \times n}(\mathbf{R})$ satisfying $S[x] \,\varepsilon\, B_\varepsilon$. Then B_ε and C_ε are open sets with compact closure, and the following limit is known to exist:

$$A_\infty(S, T) = \lim_{\varepsilon \to 0} \int_{C_\varepsilon} dx \Big/ \int_{B_\varepsilon} dt.$$

THEOREM (Siegel [4]). For $m - n \geqslant 3$,

$$(S) \qquad A(\text{genus}(S), T) = A_\infty(S, T) \lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}}.$$

4. DERIVATION OF SIEGEL'S THEOREM

Let $G = \{g \in SL(m) : S[g] = S\}$, and let $X = \{x \in M_{m \times n} : S[x] = T\}$. If $m \geqslant 4$, both $G_{\mathbf{C}}$ and $G_{\xi\mathbf{C}}$ have fundamental groups of order 2. Condition (c) of § 2 is the classical Witt theorem for (G, X). We assume that $X_{\mathbf{Q}}$ is nonempty.

We will show that (A) implies (S). This reduces Siegel's theorem to the computation of the Tamagawa number $\tau(G)$.

Let Φ_∞ = the constant function 1 on $X_{\mathbf{R}}$, and let Φ_p = the characteristic function of $X_{\mathbf{Z}_p}$ in $X_{\mathbf{Q}_p}$. Then $\Phi = \Phi_\infty \cdot \prod \Phi_p$ is the characteristic function of $X_{S_\infty} = X_{\mathbf{R}} \cdot \prod X_{\mathbf{Z}_p}$ in X_A. Because of the positive definiteness of S, Φ has compact support.

Consider the right-hand side of formula (S). Siegel has shown that there exists an algebraic gauge form dx on X such that $A_\infty(S, T) = \int_{X_{\mathbf{R}}} dx_\infty$, and

$$\lim_q \frac{A_q(S, T)}{q^{mn - (n+1)n/2}} = \prod_p \int_{X_{\mathbf{Z}_p}} dx_p,$$

where dx and dx_p are the positive measures induced on $X_{\mathbf{R}}$ and $X_{\mathbf{Q}_p}$ by dx.

It remains to identify the left-hand sides of (A) and (S). First we analyze the denominator of the left-hand side of (A). Since $\tau(G) = \tau(G_\xi) = 2$ ([5]), this denominator is $\int_{G_A/G_{\mathbf{Q}}} dg$. Now G_A admits a double coset decomposition

$$G_A = G_{S\infty}\sigma_1 G_{\mathbf{Q}} \dots G_{S\infty}\sigma_h G_{\mathbf{Q}} .$$

Then, following Tamagawa [5].

$$\begin{aligned}
\int_{G_A/G_{\mathbf{Q}}} dg &= \sum_1^h \int_{(G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} dg = \\
&= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} dg \\
&= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i)/G(\sigma_i)} dg ,
\end{aligned}$$

where $G(\sigma_i) = \sigma_i^{-1} G_{S\infty}\sigma_i \cap G_{\mathbf{Q}}$. This reduces to

$$\frac{\sum_1^h \int_{G_{S\infty}} dg}{\# G(\sigma_i)} = \int_{G_{S\infty}} dg \cdot \sum_1^h e_i .$$

A similar reduction applies to the numerator. First observe that for our choice of Φ,

$$\begin{aligned}
\sum_{x\varepsilon X_{\mathbf{Q}}} \Phi(gx) &= \#(X_{\mathbf{Q}} \cap g^{-1} X_{S\infty}) \\
&= \#(gX_{\mathbf{Q}} \cap X_{S\infty}) .
\end{aligned}$$

Then

$$\begin{aligned}
\int_{G_A/G_{\mathbf{Q}}} \sum_{x\varepsilon X_{\mathbf{Q}}} \Phi(gx)\, dg &= \int_{G_A/G_{\mathbf{Q}}} \#(gX_{\mathbf{Q}} \cap X_{S\infty})\, dg \\
&= \sum_1^h \int_{(G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} \#(gX_{\mathbf{Q}} \cap X_{S\infty})\, dg = \\
&= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i G_{\mathbf{Q}})/G_{\mathbf{Q}}} \#(\sigma_i g X_{\mathbf{Q}} \cap X_{S\infty})\, dg \\
&= \sum_1^h \int_{(\sigma_i^{-1} G_{S\infty}\sigma_i)/G(\sigma_i)} \#(\sigma_i g X_{\mathbf{Q}} \cap X_{S\infty})\, dg \\
&= \sum_1^h e_i \int_{G_{S\infty}} \#(g\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})\, dg \\
&= \sum_1^h e_i \int_{G_{S\infty}} \#(\sigma_i X_{\mathbf{Q}} \cap g^{-1} X_{S\infty})\, dg
\end{aligned}$$

$$= \sum_1^h e_i \int_{G_{S\infty}} \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})\, dg$$

$$= \int_{G_{S\infty}} dg \cdot \sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty}) .$$

The left-hand side of (A) therefore becomes

$$\frac{\int_{G_{S\infty}} dg \cdot \sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})}{\int_{G_{S\infty}} dg \cdot \sum_1^h e_i} = \frac{\sum_1^h e_i \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})}{\sum_1^h e_i} .$$

The following result completes the identification of the left-hand side of (A) with $A\big(\text{genus}\,(S), T\big)$.

PROPOSITION. $A(S_i, T) = \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty})$.

Before giving the proof, we reinterpret the matrices $S_1, \ldots, S_h$. G_A acts on the set of $\mathbf{Z}$-lattices in $\mathbf{Q}^n$ as follows: for $\sigma \,\varepsilon\, G_A$ and a lattice L, $\sigma * L$ is the unique lattice satisfying

$$(\sigma^* L) \otimes \mathbf{Z}_p = \sigma_p (L \otimes \mathbf{Z}_p) ,$$

for all p.

The matrix S defines a quadratic form on $\mathbf{Q}^n$ by $q(x) = S[x]$. Consider the lattices $\sigma_1 * \mathbf{Z}^n, \ldots, \sigma_h * \mathbf{Z}^n$. In each lattice $\sigma_i * \mathbf{Z}^n$ choose a $\mathbf{Z}$-basis, and let S_i be the matrix of q with respect to this basis. Then $S_1, \ldots, S_h$ form a complete set of representatives of the h classes in genus (S) (see [7]).

Decomposing $(SL_m)_A = (SL_m)_{S\infty} (SL_m)_{\mathbf{Q}}$ we see that each $\sigma_i \,\varepsilon\, G_A$ can be written $\sigma_i = u_i a_i$, where $u_i \,\varepsilon\, (SL_m)_{S\infty}$, $a_i \,\varepsilon\, (SL_m)_{\mathbf{Q}}$. Then

$$\sigma_i^{-1} * \mathbf{Z}^m = a_i^{-1} u_i^{-1} * \mathbf{Z}^m = a_i^{-1} * (u_i^{-1} * \mathbf{Z}^m)$$
$$= a_i^{-1} * \mathbf{Z}^m = a_i^{-1} \mathbf{Z}^m .$$

Let $w_1, \ldots, w_m$ be the standard $\mathbf{Z}$-basis of $\mathbf{Z}^m$; then $a_i^{-1} w_1, \ldots, a_i^{-1} w_m$ is a $\mathbf{Z}$-basis of $\sigma_i^{-1} * \mathbf{Z}^m$. The matrix of q with respect to this basis is

$$S_i = S[a_i^{-1}] = S[\sigma_i][a_i^{-1}] = S[\sigma_i a_i^{-1}] = S[u_i] .$$

LEMMA. Let $X_i = \{x \,\varepsilon\, M_{m \times n} : S_i[x] = T\}$. Then

(1) $(X_i)_{\mathbf{Q}} = a_i X_{\mathbf{Q}}$,

(2) $(X_i)_{S\infty} = u_i^{-1} X_{S\infty}$.

Proof of (1): Let $x \varepsilon X_A$. $a_i x \varepsilon a_i X_{\mathbf{Q}} \Leftrightarrow x$ is $\mathbf{Q}$-rational and $S[x] = T \Leftrightarrow a_i x$ is $\mathbf{Q}$-rational and $S_i[a_i x] = T \Leftrightarrow a_i x \varepsilon (X_i)_{\mathbf{Q}}$.

The proof of (2) is similar.

Now we prove the proposition.

$$\begin{aligned} A(S_i, T) &= \#(X_i)_{\mathbf{Z}} = \#((X_i)_{\mathbf{Q}} \cap (X_i)_{S\infty}) = \#(a_i X_{\mathbf{Q}} \cap u_i^{-1} X_{S\infty}) \\ &= \#(u_i a_i X_{\mathbf{Q}} \cap X_{S\infty}) = \#(\sigma_i X_{\mathbf{Q}} \cap X_{S\infty}) . \end{aligned}$$

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