

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 23 (1977)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON REPRESENTATION OF FUNCTIONS BY MEANS OF SUPERPOSITIONS AND RELATED TOPICS
Autor: Vitushkin, A. G.
Kapitel: §2. The problem of resolvents
DOI: <https://doi.org/10.5169/seals-48931>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 16.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$\xi(x, y) = \sum_{n=1}^{\infty} \frac{x^n}{n^y}$ is not a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables.

The proof of this result is based on the fact that the function $\xi(x, y)$ does not satisfy any algebraic partial differential equation, that is, an equation of the form

$$\Phi \left(\xi, \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \dots, \frac{\partial^{\mu+\lambda} \xi(x, y)}{\partial x^{\mu} \partial y^{\lambda}} \right) = 0, \quad \text{where } \Phi$$

is a polynomial with constant coefficients in the function ξ and its partial derivatives up to a certain order. At the same, it is comparatively simple to prove that any function of two variables which is a finite superposition of infinitely differentiable functions of one variable and algebraic functions of any number of variables necessarily satisfies some algebraic partial differential equation. In the same paper, Ostrowski conjectured that the function $\xi(x, y)$ is not a superposition of continuous functions of one variable and algebraic functions of any number of variables (see the theorem of Kolmogorov [9]).

§ 2. *The problem of resolvents*

Algebraic equations up to the 4-th degree inclusive are soluble by radicals, that is, the roots of these equations can be represented as functions of the coefficients in the form of a superposition of arithmetic operations and functions of one variable of the form $\sqrt[n]{t}$ ($n=2, 3$). The general equation of the 5-th degree, is insoluble by radicals, as Abel and Galois showed. But since the general equation of the 5-th degree may be reduced by algebraic substitutions to the form $x^5 + tx + 1 = 0$, containing a single parameter t , we may say that a root of the general equation of the 5-th degree is also represented as a function of the coefficients in the form of superpositions of arithmetic operations and algebraic functions of one variable. The problem of resolvents can be formulated in terms of superpositions in the following way: to find, for any number n , the smallest number k such that a root of the general equation of the n -th degree as a function of the coefficients is represented in the form of a superposition of algebraic functions of k variables. In [3] Hilbert conjectured that for $n = 6, 7, 8$ the number k is 2, 3, 4, respectively. Hilbert's result [3] for an equation of the 9-th degree was all the more unexpected: a root of the general equation of the 9-th

degree is representable as a superposition of algebraic functions of four variables. Wiman [13], generalizing Hilbert's result, proved that $k \leq n - 5$ for any $n \geq 9$. As G. N. Chebotarev [14] observed, it can be proved by the same method that $k \leq n - 6$ for $n \geq 21$ and $k \leq n - 7$ for $n \geq 121$. A number of papers by N. G. Chebotarev [15] was devoted to the problem of resolvents. However, the basic Theorem turned out to be wrong (see [16]).

In correcting Chebotarev's paper Morosov found the right statements but his proofs also were not without essential gaps [17]. Nevertheless, in spite of the mistakes the papers of Chebotarev and Morosov have had a positive influence on subsequent authors.

Arnol'd [18] and Lin [17] have shown that the function $f_n = f(z_1, \dots, z_n)$ which is the solution of the algebraic equation $f^n + z_1 f^{n-1} + z_2 f^{n-2} + \dots + z_n = 0$ for $n \geq 3$ can not be strictly represented as a superposition of entire algebraic functions of a smaller number of variables and polynomials of any number of variables. Let us recall that a function $f = f(z_1, \dots, z_k)$ is called an entire algebraic function if it satisfies an equation $f^m + p_1 f^{m-1} + \dots + p_m = 0$, where p_1, \dots, p_m are polynomials in z_1, \dots, z_k . The sentence "a function can not be strictly represented as a superposition" means in the case under consideration that every superposition representing the function must have unnecessary branches, i.e. the number of branches of any superposition must be at least $n + 1$. Using that theorem for $n = \{ 3, 4 \}$ we see that in spite of the fact that the equations of degree 3 and 4 are soluble by radicals they do not have strict representations. This explains in a sense why unnecessary roots appear when one uses Cardano's formulas.

Hovanski (see [19] and [20]) has shown that the solution of the equation $f^5 + xf^2 + yf + 1 = 0$ can not be represented by a superposition of entire algebraic functions of a single variable and polynomials in several variables. We recall that the Tschirnhaus transformation reduces the general equation of the 5-th degree to an equation with a single parameter, that is, the function of Hovanski is represented by a superposition of algebraic functions of a single variable and arithmetic operations. This counter example demonstrates that the restriction not to use the operation of division, is really strong.

We conclude the discussion of the problem of resolvents with a formulation of a well-known problem: is it possible to represent any algebraic function by means of a superposition of functions of a single variable and rational functions of any number of variables.