CHARACTERISTIC NUMBERS OF 3-MANIFOLDS [1)]

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This is a brief report on work which will be published in detail elsewhere.

All manifolds are to be closed and connected. By a *characteristic number* of a manifold M, we will mean a real valued topological invariant $\varphi(M)$ with the following multiplicative property. *If* $\varphi(M)$ *is defined, and* $\tilde{M}$ *is a k-sheeted covering manifold of* M, *then* $\varphi(\tilde{M})$ *should be defined and equal to* $k\varphi(M)$. This definition makes sense in any dimension, but we will concentrate on the 3-dimensional case.

First let us describe some examples of characteristic numbers which are defined only for very special manifolds.

We will say that a manifold H of dimension $n \geqslant 2$ is *hyperbolic* if it admits a Riemannian metric with all sectional curvatures -1. The n-dimensional volume $v(H)$ of such a manifold is a topological invariant. (In fact if $n \geqslant 3$ one can make the sharper statement that the fundamental group determines H up to isometry. Compare [Mostow].) Clearly this volume $v(H)$ satisfies the multiplicative property above, and hence is a characteristic number of H.

In the even dimensional case $n = 2m$, this volume can be computed by the generalized Gauss-Bonnet theorem, and is equal to the integer $|\chi(H)|$ multiplied by the constant $v(S^{2m})/2 = (2\pi)^m/(1\cdot 3\cdot 5 \cdots (2m-1))$. By way of contrast, in odd dimensions and in particular for $n = 3$, nothing at all is known about the number theoretic properties of these volumes $v(H)$. It may be conjectured that there exist countably many hyperbolic 3-manifolds $H_1, H_2, \ldots$ so that the real numbers $v(H_1), v(H_2), \ldots$ are linearly independent over the rational numbers. (If $v(H_1)/v(H_2)$ is irrational, it follows of course that no finite covering manifold of H_1 can be homeomorphic to a finite covering manifold of H_2.)

Similarly, if the manifold M has a Riemannian metric with all sectional curvatures $+1$, then the associated volume $v(M)$ is a topological invariant. However, in this case the sphere S^n is a finite covering manifold of M, so

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$v(M)$ is just the constant $v(S^n)$ divided by the order of the fundamental group.

R. Kulkarni has pointed out that there is a third class of closed 3-manifolds of constant curvature, namely those which possess a Lorentz metric (i.e., a pseudo-Riemannian metric of signature $+, -, -$) with constant sectional curvature $+1$.

Examples can be constructed as follows. Consider the projective special linear group $G = PSL(2, R)$ acting on the upper half-plane H^2. The Killing metric trace $(\mathrm{ad}(x)\,\mathrm{ad}(y))$ on the Lie algebra of G, multiplied by a constant factor of $-1/8$, gives rise to a left and right invariant Lorentz metric on G which has constant curvature $+1$. Choosing any discrete co-compact subgroup Γ, the quotient $M = G/\Gamma$ will be a closed 3-manifold which inherits a Lorentz metric of curvature $+1$.

To compute the Lorentz volume $v(M)$, first choose a subgroup $\tilde{\Gamma} \subset \Gamma$ of finite index k which is torsionfree. The associated k-fold covering manifold $\tilde{M} = G/\tilde{\Gamma}$ can be identified with the unit tangent bundle of the hyperbolic surface $B = H^2/\tilde{\Gamma}$. Hence the volume $v(\tilde{M})$ is proportional to the area of B, which can be calculated by the Gauss-Bonnet theorem. In this way, one sees that

$$v(\tilde{M}) = |\chi(B)|\,\pi^2/2\,.$$

It follows that the original volume $v(M)$ is equal to $|\chi(B)|\,\pi^2/2k$. *In particular,* $v(M)$ *is always a rational multiple of* $\pi^2/2$.

Furthermore this volume is a topological invariant, and hence a characteristic number of M. The following result is essentially due to [Bailey].

THEOREM 1. *Let* M *be any* 3-*manifold such that some* k-*fold covering manifold* $\tilde{M}$ *fibers as a circle bundle with Euler number* $e \neq 0$ *over a surface* B. *Then the rational number* $\chi(B)^2/k\,|e|$ *is a well defined characteristic number of* M.

In other words, this ratio does not depend on the particular choice of $\tilde{M}$ or the particular choice of fibration. In the special case $M = G/\Gamma$ considered above, where $\tilde{M}$ can be chosen as the tangent circle bundle of B with Euler number $e = \pm\,\chi(B)$, this characteristic number $\chi(B)^2/k\,|e|$ $= |\chi(B)|/k$ can be described as the Lorentz volume $v(M)$ divided by $\pi^2/2$.

The proof of Theorem 1 is based on the observation that a fibration of $\tilde{M}$ by circles gives rise to a fibration of any finite covering manifold of $\tilde{M}$ by circles. Details will be omitted.

More generally, if M is an arbitrary Lorentz 3-manifold of curvature $+1$, it would be interesting to know whether $v(M)$ is a topological invariant, or whether it is necessarily a rational multiple of $\pi^2/2$.

Now let us give an example of a characteristic number which is defined for arbitrary closed 3-manifolds M. First define an integer valued invariant $\Sigma(M)$ as follows. Let $\Sigma(M)$ be the smallest possible number of 3-simplices which can be used to triangulate M. If $\tilde{M}$ is a k-sheeted covering of M, then evidently

$$\Sigma(\tilde{M}) \leqslant k\,\Sigma(M)\,.$$

Now define

$$\sigma(M) = \inf\{\,\Sigma(\tilde{M})/k\,\}$$

taking the infimum over all finite covering manifolds of M. It follows easily that $\sigma(M)$ is a well defined, real valued characteristic number, with $0 \leqslant \sigma(M) \leqslant \Sigma(M)$.

In the special case of a hyperbolic manifold H, we will show that $\sigma(H)$ is never zero by proving the inequality

$$\sigma(H) \geqslant v(H)/v_0 > 0\,.$$

Here the constant

$$v_0 = \frac{3\sqrt{3}}{4}\left(1 - \frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{5^2} + - \ldots\right) = 1.0149416\ldots$$

is defined to be the supremum of the volumes of geodesic 3-simplices in hyperbolic space. (Compare [Coxeter].)

More generally, consider the following situation. Let M and H be oriented, with H hyperbolic.

THEOREM 2. *If there exists a map $f: M \to H$ of degree d, then* $\sigma(M) \geqslant |d|\, v(H)/v_0$.

Applying this theorem to the identity map of H, we obtain the inequality $\sigma(H) \geqslant v(H)/v_0$ mentioned above. (In the case of a non-orientable hyperbolic manifold, one must first pass to the orientable two-fold covering manifold in order to apply this argument.)

Outline proof of Theorem 2. Choose a triangulation of M with the minimum number $\Sigma(M)$ of 3-simplices. Let $f: \hat{M} \to \hat{H}$ be the induced map of universal covering spaces. After an equivariant homotopy, we may assume that $\hat{f}$ maps each simplex of $\hat{M}$ to a geodesic simplex in the hyperbolic 3-space $\hat{H}$. Hence each simplex Δ of M maps to a set $f(\Delta)$ of volume less than v_0 in H. Since a general point of H must be covered at least $|d|$ times, it follows that $\Sigma(M)\, v_0 > |d|\, v(H)$. Similarly, for a k-fold covering $\tilde{M}$ of M, it follows that

$$\Sigma(\tilde{M})\, v_0 > |kd|\, v(H)\,.$$

Dividing by $v_0 k$ and taking the infimum over all finite coverings, we obtain $\sigma(M) \geqslant |d|\, v(H)/v_0$, as required.

This result suggests the conjecture[1]) that for every map $f: H \to H'$ between oriented hyperbolic 3-manifolds the inequality

$$v(H) \geqslant v(H')\, |\deg f|$$

should be satisfied. The analogous inequality for mappings between hyperbolic surfaces was proved by [Kneser].

We are able to prove the following. Let H_0 be hyperbolic.

Theorem 3. *There exists a characteristic number* $\varphi(M)$, *defined for all* 3-*manifolds, so that* $\varphi(H_0) > 0$, *and so that*

$$\varphi(M) \geqslant |d|\, \varphi(N)$$

whenever there exists a map $M \to N$ *of degree* d *between oriented manifolds* M *and* N.

The proof will be based on the following.

Definition. A *virtual map* $M \dashrightarrow N$ will mean a pair consisting of a not necessarily connected finite covering manifold of M, with projection $p: \tilde{M} \to M$, together with a map $f: \tilde{M} \to N$. The *composition* of two virtual maps $M \dashrightarrow N \dashrightarrow P$ is a virtual map $M \dashrightarrow P$ which is readily constructed from the following diagram

$$\begin{tikzcd}
 & & \tilde{M} \times_N \tilde{N} \arrow[dl] \arrow[dr] & & \\
 & \tilde{M} \arrow[dl] \arrow[dr] & & \tilde{N} \arrow[dl] \arrow[dr] & \\
M \arrow[rr, dashed] & & N \arrow[rr, dashed] & & P
\end{tikzcd}$$

[1]) Added in proof. This conjecture has recently been proved by M. Gromov.

(Note that the product of $\tilde{M}$ and $\tilde{N}$ over N need not be connected, even if both $\tilde{M}$ and $\tilde{N}$ are connected.)

If both M and N are oriented then, giving $\tilde{M}$ the induced orientation, the *degree* of a virtual map $M \dashrightarrow N$ is defined to be the degree of $f : \tilde{M} \to N$ divided by the number of sheets of the covering (or divided by the degree of $p: \tilde{M} \to M$). One can check that the degree of a composition equals the product of the degrees.

Now consider all possible virtual maps from M to N, and let $s(M/N)$ denote the supremum of the absolute values of their degrees. Evidently

$$0 \leqslant s(M/N) \leqslant \infty .$$

This supremum definitely can be infinite, for example when the target manifold N is the sphere S^3. On the other hand, if the target manifold is hyperbolic, then the inequality

$$s(M/H) \leqslant v_0 \sigma(M)/v(H) < \infty$$

follows easily from Theorem 2. Note the inequality

$$s(M/P) \geqslant s(M/N)\, s(N/P)$$

(where the product $0.\infty$ must be interpreted as 0). In the case $M = N = P$, it follows from this inequality that $s(M/M)$ can only be either 1 or ∞. For example, $s(H/H) = 1$ but $s(S^3/S^3) = \infty$.

Proof of Theorem 3. Fixing some oriented hyperbolic manifold H_0, define

$$\varphi(M) = s(M/H_0)$$

whenever M is orientable. This is a well defined characteristic number, with $\varphi(H_0) = 1$, and with

$$\varphi(M) \geqslant s(M/N)\, \varphi(N) \geqslant |d|\, \varphi(N)$$

whenever there exists a map from M to N of degree d. The definition can be extended to a non-orientable M by setting $\varphi(M) = \varphi(\tilde{M})/2$, where $\tilde{M}$ is the orientable two-sheeted covering of M.

Other examples of characteristic numbers will be described in a more detailed manuscript, now in preparation. Some of these characteristic numbers will be defined only for oriented manifolds and will definitely depend on orientation. Others will depend only on the fundamental group of M.

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