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$$f_0(z) = \begin{cases} 0 & \text{for} \quad z \in \Delta \\ 1 & \text{for} \quad z = z_n (n = 1, 2, \ldots) . \end{cases}$$

Then let f be a continuous extension of f_0 to all of H. We now show that f is the desired function.

Let Q be a residual subset of R. Then, for each integer n, the set $Q_n = Q \cap [3n/4, 3(n+1)/4]$ is a residual subset of the closed interval $[3n/4, 3(n+1)/4]$. As a consequence of (III) for $\xi = \xi_n$, there exists a residual set of directions $\Theta_n \subset \{\theta : 3n/2 \leqslant \cot\theta \leqslant 3(n+1)/2\}$ such that, for each $\theta \in \Theta_n$, there exists a segment in Δ emanating from a point of Q_n and having the direction θ. Therefore, the set $\bigcap_{x \in Q} \Theta(x)$ is of the first category on the set $\{\theta : 3n/2 \leqslant \cot\theta \leqslant 3(n+1)/2\}$ for each integer n, and the theorem is proved.

§4. An Essential Cluster Set Example

If f is a measurable function from H to W, then the *essential cluster set* $C_e(f, x)$ *of* f *at* x is defined as the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ at x is positive for every open set U containing w; the *essential cluster set* $C_e(f, x, \theta)$ *of* f *at* x *in the direction* θ is the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ along the ray at x having direction θ is positive for every open set U containing w. As a supplement to a result of Casper Goffman and W. T. Sledd [4, Theorem 2], the present authors [1] proved the following result concerning the set

$$\Theta^*(x) = \{\theta : C_e(f, x) \subset C_e(f, x, \theta)\} \quad (x \in R) .$$

Theorem B.E.H. *If* $f : H \to W$ *is measurable, then* $\mu(\Theta^*(x)) = \pi$ *for almost every and nearly every* $x \in R$; *furthermore, if* f *is continuous, then* $\Theta^*(x)$ *is residual for almost every and nearly every* $x \in R$.

Again, a natural question to ask is whether or not, for a given function f, there exists a "large" set of directions Θ^* such that $\Theta^* \subset \Theta^*(x)$ for a "large" set of points $x \in R$. As a partial answer, we prove

Theorem 2. *There exists a continuous* $f : H \to W$ *such that the intersection* $\bigcap_{x \in Q} \Theta^*(x)$ *is* (a) *of the first category if* $Q \subset R$ *is residual, and* (b) *of measure zero if* $Q \subset R$ *is of full measure.*

Proof. Let Δ be as in the proof of Theorem 1, and let S be a closed subset of $H - \Delta$ that has metric density 1 at each $x \in R$. Let f be a continuous function on H with $f(\Delta) = \{0\}$ and $f(S) = \{1\}$. Then the proof of (a) is completely analogous to the proof of Theorem 1, and the proof of (b) follows the same line with property (IV) used in place of property (III).

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