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Let $\mathscr{I}$ be the collection of all closed intervals on $T_\xi(1/2)$ of the form $[\alpha_{nk}, \alpha_{n(k+1)}]$. Then it is easy to verify that

$$\mu(\hat{A}) = \inf \sum_{j=1}^{\infty} \mu(I_j),$$

where the inf is taken over all sequences $\{I_j\}$ of intervals in $\mathscr{I}$ covering $\hat{A}$.

Let $\varepsilon > 0$. Then there is a sequence $\{I_j\}$ of intervals in $\mathscr{I}$ that cover $\hat{A}$ such that

$$\sum_{j=1}^{\infty} \mu(I_j) < \varepsilon .$$

For each index j, there exist integers n_j and k_j such that

$$I_j = T_{n_j k_j}(1/2);$$

hence, in view of (B) in §1, we have

$$\Lambda(\hat{A}) \subset \bigcup_{j=1}^{\infty} \operatorname{proj}[T_{n_j k_j}] .$$

Furthermore, combining (A) and (B) of §1, we obtain

$$\mu(\operatorname{proj}[T_{n_j k_j}]) = 2\mu(I_j) \quad (j=1, 2, \ldots).$$

Therefore,

$$\mu^*(\Lambda(\hat{A})) \leqslant \sum_{j=1}^{\infty} \mu(\operatorname{proj}[T_{n_j k_j}]) = 2 \sum_{j=1}^{\infty} \mu(I_j) < 2\varepsilon ,$$

and property (IV) is proved.

§3. Proof of Theorem 1

For each integer n, let $\xi_n = 1 + 3n/2$ and set

$$\Delta_n = K_{\xi_n} \cap \{(x, y): \xi_n/2 \leqslant x \leqslant \xi_n/2 + 3/4 \quad \text{and} \quad 1/2 < y \leqslant 1\}.$$

Then set

$$\Delta_n^* = \{z - (1+i)/2: z \in \Delta_n\} \quad (i = \sqrt{-1}),$$

and define the set

$$\Delta = \cup \{\Delta_n^*: n = 0, \pm 1, \pm 2, \ldots\}.$$

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $H - \Delta$ whose derived set is R. Define the function f_0 on $\Delta \cup \{z_n\}_{n=1}^{\infty}$ by

$$f_0(z) = \begin{cases} 0 & \text{for} \quad z \in \Delta \\ 1 & \text{for} \quad z = z_n (n = 1, 2, \ldots) . \end{cases}$$

Then let f be a continuous extension of f_0 to all of H. We now show that f is the desired function.

Let Q be a residual subset of R. Then, for each integer n, the set $Q_n = Q \cap [3n/4, 3(n+1)/4]$ is a residual subset of the closed interval $[3n/4, 3(n+1)/4]$. As a consequence of (III) for $\xi = \xi_n$, there exists a residual set of directions $\Theta_n \subset \{ \theta: 3n/2 \leqslant \cot \theta \leqslant 3(n+1)/2 \}$ such that, for each $\theta \in \Theta_n$, there exists a segment in Δ emanating from a point of Q_n and having the direction θ. Therefore, the set $\bigcap_{x \in Q} \Theta(x)$ is of the first category on the set $\{ \theta: 3n/2 \leqslant \cot \theta \leqslant 3(n+1)/2 \}$ for each integer n, and the theorem is proved.

§4. An Essential Cluster Set Example

If f is a measurable function from H to W, then the *essential cluster set* $C_e(f, x)$ *of* f *at* x is defined as the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ at x is positive for every open set U containing w; the *essential cluster set* $C_e(f, x, \theta)$ *of* f *at* x *in the direction* θ is the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ along the ray at x having direction θ is positive for every open set U containing w. As a supplement to a result of Casper Goffman and W. T. Sledd [4, Theorem 2], the present authors [1] proved the following result concerning the set

$$\Theta^*(x) = \{ \theta: C_e(f, x) \subset C_e(f, x, \theta) \} \quad (x \in R) .$$

Theorem B.E.H. *If* $f: H \to W$ *is measurable, then* $\mu(\Theta^*(x)) = \pi$ *for almost every and nearly every* $x \in R$; *furthermore, if* f *is continuous, then* $\Theta^*(x)$ *is residual for almost every and nearly every* $x \in R$.

Again, a natural question to ask is whether or not, for a given function f, there exists a "large" set of directions Θ^* such that $\Theta^* \subset \Theta^*(x)$ for a "large" set of points $x \in R$. As a partial answer, we prove

Theorem 2. *There exists a continuous* $f: H \to W$ *such that the intersection* $\bigcap_{x \in Q} \Theta^*(x)$ *is* (a) *of the first category if* $Q \subset R$ *is residual, and* (b) *of measure zero if* $Q \subset R$ *is of full measure.*