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Let \mathcal{I} be the collection of all closed intervals on $T_\xi(1/2)$ of the form $[\alpha_{nk}, \alpha_{n(k+1)}]$. Then it is easy to verify that

$$\mu(\hat{A}) = \inf \sum_{j=1}^{\infty} \mu(I_j),$$

where the inf is taken over all sequences $\{I_j\}$ of intervals in \mathcal{I} covering \hat{A} .

Let $\varepsilon > 0$. Then there is a sequence $\{I_j\}$ of intervals in \mathcal{I} that cover \hat{A} such that

$$\sum_{j=1}^{\infty} \mu(I_j) < \varepsilon.$$

For each index j , there exist integers n_j and k_j such that

$$I_j = T_{n_j k_j}(1/2);$$

hence, in view of (B) in §1, we have

$$\Lambda(\hat{A}) \subset \bigcup_{j=1}^{\infty} \text{proj}[T_{n_j k_j}].$$

Furthermore, combining (A) and (B) of §1, we obtain

$$\mu(\text{proj}[T_{n_j k_j}]) = 2\mu(I_j) \quad (j = 1, 2, \dots).$$

Therefore,

$$\mu^*(\Lambda(\hat{A})) \leq \sum_{j=1}^{\infty} \mu(\text{proj}[T_{n_j k_j}]) = 2 \sum_{j=1}^{\infty} \mu(I_j) < 2\varepsilon,$$

and property (IV) is proved.

§3. PROOF OF THEOREM 1

For each integer n , let $\xi_n = 1 + 3n/2$ and set

$$\Delta_n = K_{\xi_n} \cap \{(x, y): \xi_n/2 \leq x \leq \xi_n/2 + 3/4 \text{ and } 1/2 < y \leq 1\}.$$

Then set

$$\Delta_n^* = \{z - (1+i)/2: z \in \Delta_n\} \quad (i = \sqrt{-1}),$$

and define the set

$$\Delta = \cup \{\Delta_n^*: n = 0, \pm 1, \pm 2, \dots\}.$$

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $H - \Delta$ whose derived set is R . Define the function f_0 on $\Delta \cup \{z_n\}_{n=1}^{\infty}$ by

$$f_0(z) = \begin{cases} 0 & \text{for } z \in \Delta \\ 1 & \text{for } z = z_n (n=1, 2, \dots). \end{cases}$$

Then let f be a continuous extension of f_0 to all of H . We now show that f is the desired function.

Let Q be a residual subset of R . Then, for each integer n , the set $Q_n = Q \cap [3n/4, 3(n+1)/4]$ is a residual subset of the closed interval $[3n/4, 3(n+1)/4]$. As a consequence of (III) for $\xi = \xi_n$, there exists a residual set of directions $\Theta_n \subset \{\theta: 3n/2 \leq \cot \theta \leq 3(n+1)/2\}$ such that, for each $\theta \in \Theta_n$, there exists a segment in Δ emanating from a point of Q_n and having the direction θ . Therefore, the set $\bigcap_{x \in Q} \Theta(x)$ is of the first category on the set $\{\theta: 3n/2 \leq \cot \theta \leq 3(n+1)/2\}$ for each integer n , and the theorem is proved.

§4. AN ESSENTIAL CLUSTER SET EXAMPLE

If f is a measurable function from H to W , then the *essential cluster set* $C_e(f, x)$ of f at x is defined as the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ at x is positive for every open set U containing w ; the *essential cluster set* $C_e(f, x, \theta)$ of f at x in the direction θ is the set of all values $w \in W$ for which the upper density of $f^{-1}(U)$ along the ray at x having direction θ is positive for every open set U containing w . As a supplement to a result of Casper Goffman and W. T. Sledd [4, Theorem 2], the present authors [1] proved the following result concerning the set

$$\Theta^*(x) = \{\theta: C_e(f, x) \subset C_e(f, x, \theta)\} \quad (x \in R).$$

THEOREM B.E.H. *If $f: H \rightarrow W$ is measurable, then $\mu(\Theta^*(x)) = \pi$ for almost every and nearly every $x \in R$; furthermore, if f is continuous, then $\Theta^*(x)$ is residual for almost every and nearly every $x \in R$.*

Again, a natural question to ask is whether or not, for a given function f , there exists a “large” set of directions Θ^* such that $\Theta^* \subset \Theta^*(x)$ for a “large” set of points $x \in R$. As a partial answer, we prove

THEOREM 2. *There exists a continuous $f: H \rightarrow W$ such that the intersection $\bigcap_{x \in Q} \Theta^*(x)$ is (a) of the first category if $Q \subset R$ is residual, and (b) of measure zero if $Q \subset R$ is of full measure.*