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§2. KAHANE'S SET K_ξ .

Let ξ be a real number, and let T_ξ be the $(1/2)$ -trapezoid with bases

$$L = \{(x, 0): 0 \leq x \leq 1\} \quad \text{and} \quad L' = \{(x', 1): \xi \leq x' \leq \xi + 1/2\}.$$

Let T_{11}, T_{12}, T_{13} , and T_{14} be the four descendants of T_ξ labeled in such a way that the points

$$\alpha_{1k} = \min \{x: (x, 1/2) \in T_{1k}\}$$

satisfy $\alpha_{1k} < \alpha_{1(k+1)}$ for $k = 1, 2$, and 3 . Then let T_{2k} ($k = 1, 2, \dots, 4^2$) be the 4^2 descendants of the trapezoids T_{1k} ($k = 1, 2, 3, 4$) labeled in such a way that the points

$$\alpha_{2k} = \min \{x: (x, 1/2) \in T_{2k}\}$$

satisfy $\alpha_{2k} < \alpha_{2(k+1)}$ for $k = 1, 2, \dots, 4^2 - 1$. Continuing inductively we arrive at the collection of $(1/2)$ -trapezoids

$$T_{nk} \quad \text{for} \quad n = 1, 2, \dots \quad \text{and} \quad k = 1, 2, \dots, 4^n$$

where the T_{nk} ($k = 1, 2, \dots, 4^n$) are the descendants of the $T_{(n-1)k}$ ($k = 1, 2, \dots, 4^{n-1}$) and are labeled in such a way that the points

$$\alpha_{nk} = \min \{x: (x, 1/2) \in T_{nk}\}$$

satisfy $\alpha_{nk} < \alpha_{n(k+1)}$ for $k = 1, 2, \dots, 4^n - 1$.

If for each n we set

$$\alpha_{n(4^n+1)} = \max \{x: (x, 1/2) \in T_{n4^n}\},$$

then it follows from (A) in §1 that both

$$T_{nk}(1/2) = \{(x, 1/2): \alpha_{nk} \leq x \leq \alpha_{n(k+1)}\}$$

for each $k = 1, 2, \dots, 4^n$, and for each n

$$T_\xi(1/2) = \bigcup_{k=1}^{4^n} T_{nk}(1/2).$$

Now, for each n , let $K_n = \bigcup_{k=1}^{4^n} T_{nk}$ and define the set

$$K_\xi = \bigcap_{n=1}^{\infty} K_n.$$

The following properties of the set K_ξ were established by Kahane [5]:

(I) K_ξ is a compact set of 2-dimensional measure zero.

(II) Let any segment τ that extends from one base of T_ξ to the other be called "admissible". Then the following two properties hold:

- (i) There are at most two admissible line segments in K_ξ that pass through any given point $z \in T_\xi(1/2)$; in fact, precisely one such segment passes through each $z \in T_\xi(1/2)$ that is not one of the α_{nk} .
- (ii) For each $\lambda \in [\xi - 1, \xi + 1/2] = \text{proj}[T_\xi]$, there exists at least one point $z \in T_\xi(1/2)$ through which there passes an admissible line segment τ in K_ξ with $\text{proj}[\tau] = \lambda$; in fact, for all but countably many such λ , there exists only one such z .

We now introduce a set-valued function Λ defined on the subsets A of $T_\xi(1/2)$ as follows:

$$\Lambda(A) = \{ \text{proj}[\tau] : \tau \subset K_\xi \text{ is an admissible segment with } A \cap \tau \neq \phi \}.$$

Then K_ξ has the two additional properties:

(III) $\Lambda(A)$ is nowhere dense whenever A is.

(IV) $\Lambda(A)$ is of linear (Lebesgue) measure zero whenever A is.

Proof of (III). Let A be a nowhere dense subset of $T_\xi(1/2)$, and suppose $\Lambda(A)$ is dense on some subinterval I of the interval $\text{proj}[T_\xi]$. Then, according to (B) in §1, there exist integers n and k such that $\Lambda(A)$ is dense on $\text{proj}[T_{nk}]$. Since A is nowhere dense, there exist integers n' and k' such that $T_{n'k'}(1/2) \subset T_{nk}(1/2)$ and $A \cap T_{n'k'} = \phi$. By (B) of §1, it follows that

$$\Lambda(A) \cap \text{int}(\text{proj}[T_{n'k'}]) = \phi$$

(int \equiv interior). This contradicts the fact that $\Lambda(A)$ is dense on $\text{proj}[T_{nk}]$, and property (III) is proved.

For the remainder of this paper, we use $\mu(A)$ and $\mu^*(A)$ to denote respectively the Lebesgue measure and the Lebesgue outer measure of the linear set A .

Proof of (IV). Let $A \subset T_\xi(1/2)$ and suppose $\mu(A) = 0$. If we set

$$\hat{A} = A - \{ \alpha_{nk} : n = 1, 2, \dots; k = 1, 2, \dots, 4^n + 1 \},$$

then in view of II (i) it suffices to show that the corresponding set $\Lambda(\hat{A})$ has linear measure zero.

Let \mathcal{I} be the collection of all closed intervals on $T_\xi(1/2)$ of the form $[\alpha_{nk}, \alpha_{n(k+1)}]$. Then it is easy to verify that

$$\mu(\hat{A}) = \inf \sum_{j=1}^{\infty} \mu(I_j),$$

where the inf is taken over all sequences $\{I_j\}$ of intervals in \mathcal{I} covering \hat{A} .

Let $\varepsilon > 0$. Then there is a sequence $\{I_j\}$ of intervals in \mathcal{I} that cover \hat{A} such that

$$\sum_{j=1}^{\infty} \mu(I_j) < \varepsilon.$$

For each index j , there exist integers n_j and k_j such that

$$I_j = T_{n_j k_j}(1/2);$$

hence, in view of (B) in §1, we have

$$\Lambda(\hat{A}) \subset \bigcup_{j=1}^{\infty} \text{proj}[T_{n_j k_j}].$$

Furthermore, combining (A) and (B) of §1, we obtain

$$\mu(\text{proj}[T_{n_j k_j}]) = 2\mu(I_j) \quad (j = 1, 2, \dots).$$

Therefore,

$$\mu^*(\Lambda(\hat{A})) \leq \sum_{j=1}^{\infty} \mu(\text{proj}[T_{n_j k_j}]) = 2 \sum_{j=1}^{\infty} \mu(I_j) < 2\varepsilon,$$

and property (IV) is proved.

§3. PROOF OF THEOREM 1

For each integer n , let $\xi_n = 1 + 3n/2$ and set

$$\Delta_n = K_{\xi_n} \cap \{(x, y): \xi_n/2 \leq x \leq \xi_n/2 + 3/4 \text{ and } 1/2 < y \leq 1\}.$$

Then set

$$\Delta_n^* = \{z - (1+i)/2: z \in \Delta_n\} \quad (i = \sqrt{-1}),$$

and define the set

$$\Delta = \cup \{\Delta_n^*: n = 0, \pm 1, \pm 2, \dots\}.$$

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of points in $H - \Delta$ whose derived set is R . Define the function f_0 on $\Delta \cup \{z_n\}_{n=1}^{\infty}$ by