

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 22 (1976)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** FINITE GEOMETRIES IN THE THEORY OF THETA CHARACTERISTICS  
**Autor:** Rivano, Neantro Saavedra  
**Kapitel:** §4 Basis and fundamental sets  
**DOI:** <https://doi.org/10.5169/seals-48185>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 23.01.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

3.7 The following construction would help in developing the case where  $|X|$  is even along the lines of 3.2-3.5, which I won't do. Let  $X$  be of odd order  $|X| = 2g + 1$ , and define  $X' = X \amalg \{X\}$ , thus  $|X'| = 2g + 2$ . We have a natural linear map

$$2^X \rightarrow 2^{X'}$$

and this is compatible with  $p, e, Q, q_0$ . Composing this with the passage to the quotient, I have a linear isomorphism

$$2^X \rightarrow P_2(X'),$$

and by compatibility with  $p, p'$ , isomorphisms

$$\begin{aligned} 2_+^X &\rightarrow P_2^+(X') \\ 2_-^X &\rightarrow P_2^-(X'). \end{aligned}$$

The first is compatible with  $e, e'$ , and with the canonical quadratic forms if  $g$  is odd. The second is compatible with  $Q, Q'$  if  $g$  is even.

#### § 4 BASIS AND FUNDAMENTAL SETS

4.1 *Normal basis.* Let  $(J, e)$  be a symplectic pair. A *normal basis* for  $(J, e)$  is a basis  $(x_i)_{i \in I}$  for  $J$  with the property that  $e(x_i, x_j) = 1$  for  $i \neq j$ , the set of ordered normal basis (i.e. for  $I = \{1, \dots, 2g\}$  if  $2g = \dim J$ ) will be denoted  $ONB(J, e)$ . The symplectic group  $Sp(J, e)$  clearly acts on  $ONB(J, e)$  and it does it simply transitively, because if two ordered normal bases for  $(J, e)$  are given, the unique linear automorphism transforming one into the other is obviously symplectic.

I have not yet shown that the set  $ONB(J, e)$  is non-empty, this we will see as a consequence of the following construction, that relates symplectic basis (0.1) with normal basis. The set  $SB(J, e)$  of symplectic basis is a torsor over  $Sp(J, e)$ , thus if  $ONB(J, e)$  is non-empty, both torsors should be isomorphic and indeed there would be as many isomorphisms as elements in the group  $Sp(J, e)$ . What I proceed to exhibit now is a definite isomorphism

$$\alpha: SB(J, e) \rightarrow ONB(J, e)$$

with inverse  $\beta$ . If

$$x \in SB(J, e), x = (x_1, \dots, x_g, x'_1, \dots, x'_g)$$

let's put  $y = \alpha(x)$ , then by definition

$$\begin{aligned} y_{2k-1} &= x_1 + \dots + x_k + x'_1 + \dots + x'_{k-1} \\ y_{2k} &= x_1 + \dots + x_{k-1} + x'_1 + \dots + x'_k \quad k = 1, \dots, g. \end{aligned}$$

As for the inverse, if  $y \in ONB(J, e)$ , and  $x = \beta(y)$ , then one gets from the definition of  $\alpha$

$$\begin{aligned} x_k &= y_1 + \dots + y_{2k-2} + y_{2k-1} \\ x'_k &= y_1 + \dots + y_{2k-2} + y_{2k} \quad k = 1, \dots, g. \end{aligned}$$

It is clear from this definition that  $\alpha$  is compatible with the actions of  $Sp(J, e)$  on both sets.

**4.2 Azygetic sets.** Let  $(S, Q)$  be a symplectic torsor over a symplectic pair  $(J, e)$ . A subset  $A \subset S$  is *azygetic* if for any three different elements  $s_1, s_2, s_3 \in A$  one has  $Q(s_1) + Q(s_2) + Q(s_3) + Q(s_1 + s_2 + s_3) = 1$ , or equivalently if  $e(s_1, +s_2, s_1 + s_3) = 1$ .  $A$  is *homogeneous* if  $Q$  is constant on it, i.e. if either  $A \subset S^+$  or  $A \subset S^-$ . And the subset  $A$  is *linearly independent* if for some (or equivalently, for any)  $s \in A$ , the subset  $s + (A - \{s\}) \subset J$  is linearly independent, or equivalently if  $A + A$  spans a subspace of  $J$  of dimension  $|A| - 1$ .

Let  $A$  be an azygetic subset,  $s \in A$ , and let  $B = s + (A - \{s\})$ , I will show that the only possible linear relation on  $B$  is  $\sum_{x \in B} x = 0$ . Indeed, if  $\sum \lambda_x x = 0$  is such a relation, for any  $y \in B$ , one has

$$\begin{aligned} 0 &= e(y, \sum_x \lambda_x x) = \sum_x \lambda_x e(y, x) = \sum_{\substack{x \in B \\ x \neq y}} \lambda_x \\ \sum_{x \neq y} \lambda_x &= 0 \end{aligned}$$

Adding these equations for any  $y, y' \in B$ , one concludes that  $\lambda_y = \lambda_{y'}$ , which was to be shown. As a consequence of this, it follows that any azygetic subset of odd order is linearly independent, and that an azygetic subset has at most  $2g + 2$  elements. It is easy to verify that if  $A$  is an azygetic subset of odd order and if  $s = \sum_{t \in A} t$ ,  $A \cup \{s\}$  is still azygetic.

**4.3 Basis for symplectic torsors.** A *basis* for a symplectic torsor  $(S, Q)$  over  $(J, e)$  is a maximal homogeneous, linearly independent, azygetic subset of  $S$ . A basis has exactly  $2g + 1$  elements, where  $g$  is the genus of  $(S, Q)$ . This comes from the fact that any symplectic torsor is isomorphic to one of the form  $(S_X, Q_X)$  constructed in § 3 because of the uniqueness result in 1.4, that for  $S_X$ ,  $X \subset S_X$  is clearly a basis with  $2g + 1$  elements, and that a linearly independent subset can have at most  $2g + 1$  elements.

The set of ordered basis for  $(S, Q)$  will be denoted by  $OB(S, Q)$ , the group  $Sp(S, Q)$  acts on it.

The following construction is fundamental. Let  $X \subset S$  be a basis, we have then a map

$$F_X: 2^X \rightarrow E(S)$$

(cf. 1.5.a) for the definition of  $E(S)$ ), defined by

$$F_X(A) = \sum_{s \in A} s$$

It is clear that  $F_X$  is a group homomorphism, that sends subsets of  $X$  of even (resp. odd) order into  $J$  (resp.  $S$ ), thereby inducing a linear homomorphism

$$\sigma_X: 2_+^X \rightarrow J$$

and a map compatible with the respective group actions

$$f_X: 2_-^X \rightarrow S.$$

To proceed further, let's choose a total order on  $X$ ,  $X = \{s_0, \dots, s_{2g}\}$ . Then, the  $X_i = \{s_0, s_i\}$  (resp.  $x_i = s_0 + s_i$ ) for  $i = 1, \dots, 2g$  constitute an ordered normal basis for  $2_+^X$  (resp.  $J$ ), and as  $\sigma_X(X_i) = x_i$  we have that  $\sigma_X$  is a symplectic isomorphism. It follows that  $f_x$  is a bijection, and indeed  $f_x$  defines an isomorphism of symplectic torsors between  $(S_X, Q_X)$  and  $(S, Q)$ . To see this, we have to prove that if  $A, A' \subset X$  are such that  $|A| \equiv |A'| \pmod{4}$ , then

$$Q\left(\sum_{s \in A} s\right) = Q\left(\sum_{s \in A'} s\right).$$

We know that  $Q$  is constant on  $X$ , and the condition on  $X$  of being azygetic means that for any three different  $s_1, s_2, s_3 \in X$ ,  $Q(s_1 + s_2 + s_3)$  is different from the value of  $Q$  on  $X$ . From this remark, the fact to be proved follows easily by induction and using the defining property (1.1.1) of symplectic torsors. For example, if  $|A| = 5$ , and we order  $A = \{s_1, \dots, s_5\}$ , we have

$$Q(\Sigma s_1) + Q(s_1) = Q(s_1 + s_2 + s_3) + Q(s_1 + s_4 + s_5)$$

because  $e(s_2 + s_3, s_4 + s_5) = 0$ , thus

$$Q(s_1) = Q(\Sigma s_i).$$

Summing up: starting from a basis  $X \subset S$ , one gets an isomorphism of symplectic pairs

$$\sigma_X: (J_X, e_X) \xrightarrow{\sim} (J, e)$$

underlying an isomorphism of symplectic torsors

$$f_X: (S_X, Q_X) \simeq (S, Q).$$

As a consequence of this, we have that a basis is necessarily contained in  $S^+$  for  $g \equiv 0, 1 \pmod{4}$ , in  $S^-$  for  $g \equiv 2, 3 \pmod{4}$  (cf. 3.2.1).

**4.4 PROPOSITION.** *The set  $OB(S, Q)$  of ordered basis for a symplectic torsor  $(S, Q)$  is a torsor over the group  $Sp(S, Q)$ . Moreover, the map*

$$OB(S, Q) \rightarrow ONB(J, e)$$

*defined by*

$$(s_i)_{0 \leq i \leq 2g} \mapsto (s_0 + s_i)_{1 \leq i \leq 2g}$$

*is an isomorphism of torsors over  $Sp(S, Q) \simeq Sp(J, e)$ .*

**4.4.1 Proof.** The map defined above is clearly compatible with the actions of  $Sp(S, Q)$ ,  $Sp(J, e)$  and the isomorphism between these groups described in 1.4. To prove the proposition, it is enough to show that this map is bijective. As  $OB(S, Q)$  is non-empty and  $ONB(J, e)$  is a torsor, this map is onto. It is injective too, because starting from the  $x_i = s_0 + s_i$  I can recover the  $s_i$  in the following way. If  $s = \sum_{0 \leq i \leq 2g} s_i$ , by the identification  $S \simeq Q(J, e)$  in 1.5,  $s$  corresponds to the unique quadratic form  $q_s$  on  $J$  whose value on each of the  $x_i$  is 1 as it can be easily seen, thus  $s$  can be defined in terms of the  $x_i$ ; but then

$$s_i = s + \sum_{j \neq i} x_j \quad (0 \leq i \leq 2g, 1 \leq j \leq 2g).$$

**4.5 Fundamental sets.** A *fundamental set* for a symplectic torsor  $(S, Q)$  is a maximal azygetic subset  $F \subset S$ . Any basis  $X$  for  $S$  defines a fundamental set, it suffices to put  $F_X = X \cup \{s_X\}$ , where  $s_X = \sum_{s \in X} s$ . Also, if  $F$  is a fundamental set and if  $x \in J$ ,  $x + F$  is a fundamental set too, as it is easily seen. In fact, for any fundamental set  $F$ , there exists a basis  $X$  and an  $x \in J$  such that  $F = x + F_X$ . Let  $F = \{t_0, \dots, t_{2g+1}\}$  be an ordering of  $F$ , it is clear that if

$$x_i = t_0 + t_i \quad (1 \leq i \leq 2g+1),$$

the  $x_i$  for  $1 \leq i \leq 2g$  constitute a normal basis for  $J$ , thus there exists a unique ordered basis  $X = \{s_0, \dots, s_{2g}\}$  for  $S$  such that  $x_i = s_0 + s_i$  (4.4). Then, if  $x = s_0 + t_0$ , we have  $t_i = x + s_X$ , because  $\sum t_i = 0$  and  $s_X = \sum s_i$ .

Observe that a fundamental set arising from a basis is homogeneous iff  $g$  is even. Indeed, it is homogeneous iff  $2g + 1 \equiv 1 \pmod{4}$ , i.e. iff  $g$  is even.

It follows from the last part of prop. 3.2.1 that, in this case, the number of odd characteristics in the fundamental sets is congruent to  $g \pmod{4}$ . We will see that this is a general fact.

4.5.1 PROPOSITION. *Let  $O(F)$  be the number of odd characteristics in a fundamental set  $F$ . Then  $O(F) \equiv g \pmod{4}$ . Conversely, for any  $l \equiv g \pmod{4}$ , and  $l \leq 2g + 2$ , there are fundamental sets  $F$  with  $O(F) = l$ .*

4.5.2 Proof. We may safely restrict ourselves to the case where the symplectic torsor is  $S_X$  with its standard basis  $X$ , and  $F = \{A\} + (X \cup \{X\})$  where  $A \subset X$  is of even order  $|A| = 2k$  (cf. 4.3). Then, in  $F$  there are  $2k$  characteristics corresponding to subsets of  $X$  with  $2k - 1$  elements,  $2(g - k) + 1$  characteristics with  $2k + 1$  elements, and 1 characteristic with  $2(g - k) + 1$  elements, namely the ones obtained adding  $A$  to respectively the characteristics of the form  $\{s\}$  ( $s \in A$ ),  $\{s\}$  ( $s \notin A$ ),  $X$ . When  $g$  is even the second and third types have the same parity; when  $g$  is odd the first and third types have the same parity. From these remarks, it is easy to see that the number of elements of the same parity in  $F$  and  $X \cup \{X\}$  are congruent mod 4, and that with this only restriction, this number can be arbitrary for  $F$  by conveniently choosing  $A$ . The proposition follows from this and from what was said just before its statement.

4.5.3 In Coble [1], additional material on fundamental sets may be found.

## REFERENCES

- [1] COBLE, A. An application of finite geometry to the characteristic theory of the odd and even theta functions. *Trans. A.M.S.* 7 (1906), pp. 241-276.
- [2] FAY, J. Theta functions on Riemann surfaces. *Lecture Notes in Mathematics*, No. 352, Springer Verlag (1973).
- [3] IGUSA, J. *Theta Functions*. Springer Verlag (1972).
- [4] MUMFORD, D. On the equations defining Abelian varieties I. *Invent. Math.* 1 (1966), pp. 287-354.
- [5] ——— Theta characteristics of an algebraic curve. *Ann. Scient. ENS* 4 (1971), pp. 181-192.
- [6] WEBER, H. *Lehrbuch der algebra*, Band 2, Braunschweig (1896).

(Reçu le 20 février 1976)

Neantro Saavedra Rivano

Dept. de Matemática  
Universidad Simón Bolívar  
Apartado 5354  
Caracas, Venezuela