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2.7 COROLLARY. Let (S, Q), (S', Q') be symplectic torsors of genus g over (J, e), (J', e'), and let $\Sigma = S^{\pm}, \Sigma' = S'^{\pm}$. Then, there are canonical bijections

$$\operatorname{Isom}\left((J, e), (J', e')\right) \simeq \operatorname{Isom}\left((S, Q), (S', Q')\right)$$
$$\simeq \operatorname{Isom}\left((\Sigma, \Sigma_{(4)}), (\Sigma', \Sigma'_{(4)})\right).$$

In particular, there are group isomorphisms

$$\operatorname{Sp}(J, e) \simeq \operatorname{Sp}(S, Q) \simeq \operatorname{Aut}(\Sigma, \Sigma_{(4)}).$$

§ 3 Symplectic torsors defined by finite sets

In this paragraph, X will be a finite set.

3.1 The basic construction. Starting from X one has

a) The set 2^{X} of subsets of X, with the operation of symmetric difference: $A + B = A \cup B - A \cap B$ $A, B \in 2^{X}$

b) A map $p: 2^X \to \mathbb{Z}/2\mathbb{Z}$ defined by $p(A) = |A|(2) \quad A \in 2^X$ c) A map $e: 2^X \times 2^X \to \mathbb{Z}/2\mathbb{Z}$ defined by $e(A, B) = |A \cap B|(2) \quad A, B \in 2^X$ d) A map $Q: 2^X_- \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$Q(B) = \frac{|B| + 1}{2}(2) \quad B \in 2^{X}_{-}$$

where $2_{-}^{X} = p^{-1}(1)$ is the set of subsets of odd order of X.

e) A map $q_o = 2^X_+ \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by

$$q_0(A) = \frac{|A|}{2}(2) \quad A \in 2^X_+$$

where $2_{+}^{X} = p^{-1}(0)$.

Then, it is easily verified that

- α) 2^x is a vector space over Z/2Z, of dimension | X |.
- β) p is linear

 γ) *e* is bilinear

 δ) Q has the following property (compare 1.1.1)

Q(B) + Q(A+B) + Q(A'+B) + Q(A+A'+B) = e(A, A')whenever $B \in 2^{X}_{-}, A, A' \in 2^{X}_{+}$

 ε) q_o is a quadratic form inducing the restriction of e to 2^{X}_{+} .

In the proof of these, one uses the following identity

 $|A + B| = |A| + |B| - 2 |A \cap B| \quad A, B \in 2^X.$

3.2 Let's assume in the following three sections that X is of odd order, |X| = 2g + 1.

3.2.1 PROPOSITION. The bilinear form e on 2^X_+ is alternate and nondegenerate. If 2^X_+ acts on 2^X_- by translations, $(2^X_-, Q)$ is a symplectic torsor over $(2^X_+, e)$ which is even for $g \equiv 2, 3$ (4) and odd for $g \equiv 0, 1$ (4).

3.2.2 *Proof.* It is clear that e is alternate on 2_+^X . It is also non degenerate, because if $A \in 2_+^X$, $A \neq \phi$, let $x \in A$; then $A' = (X - A) \cup \{x\}$ is of even order, and e(A, A') = 1. It is also clear that $(2_-^X, Q)$ is a symplectic torsor over $(2_+^X, e)$ (because of 3.1 δ) and the definition of symplectic torsor.

To find out when this torsor is even or odd, we first observe that it is clearly odd for g = 0, 1 (look at it), then apply descending induction using the following fact (to be proved below). Let's call ε_g the type of the torsor corresponding to an X with |X| = 2g + 1 (and $g \ge 2$), thus $\varepsilon_g = \pm 1$; then $\varepsilon_g = \varepsilon_{g-1}$ if g is odd, and $\varepsilon_g = -\varepsilon_{g-1}$ if g is even.

Proof of this fact: take a fixed $A_o \subset X$ of order two. The set of $B \in 2_-^X$ such that $Q(B) = Q(A_o + B) = 0$ (recall that Q(B) = 0 means that $|B| \equiv 1$ (4)) has cardinality $2^{g-1} (2^{g-1} + \varepsilon_g)$ by definition of ε_g and proposition 2.1.1. But clearly this number is also twice the cardinality of the set of subsets C of $X - A_o$ such that $|C| \equiv 2g - 1$ (4) (in fact any such Bdefines a C by $C = X - (A_o \cup B)$ and this map is two-fold) and the number of these is $2^{g-2} (2^{g-1} + \varepsilon_{g-1})$ or $2^{g-2} (2^{g-1} - \varepsilon_{g-1})$ according to 2g - 1 $\equiv 1$ (4) or $2g - 1 \equiv 3$ (4), i.e. g odd or even. This proves the fact and completes the proof of the proposition.

3.3 If Q is odd, let us agree to modify Q in the way described in 1.1 to obtain an even torsor \overline{Q} . With this convention, the following notation will be adopted:

$$J_X = 2^X_+ \quad e_X = e$$
$$S_X = 2^X_- \quad Q_X = Q$$

or \overline{Q} according to the value of $g \mod 4$.

The identification $S_X \simeq Q(J_X, e_X)$ in 1.4 may be made explicit: if $B \in S_X$, B becomes the following quadratic form

$$B(A) = |A \cap B| + \frac{|A|}{2}(2).$$

Let's now make explicit the condition for a triple (B_1, B_2, B_3) of elements of either S_X^+ or S_X^- to be a *triplet* (2.3). This means that

$$Q_X(\Sigma B_i) = \Sigma Q_X(B_i),$$

and this is equivalent to

$$\sum_{i < j} |B_i \cap B_j| \equiv 1 (2) ,$$

or still to

$$|\cup B| \equiv |\cap B_i|(2).$$

3.4 The quadratic form q_o on J_X singled out in 3.1 e) corresponds through the identification $Q(J_X, e_X) = S_X$ to X itself. As $Q(X) \equiv g + 1$ (2), it results from the last part of 3.2.1 that the Arf invariant of q_o is 0 for $g \equiv 0, 3$ (4), 1 for $g \equiv 1, 2$ (4). In other words, $q_o \in S_X^+$ for $g \equiv 0, 3$ (4), $q_o \in S_X^-$ for $g \equiv 1, 2$ (4).

3.5 Let's assume in this and the next sections that X is of even order, |X| = 2g + 2. Then, the linear map p passes to the quotient $2^X / \{0, X\}$. This quotient identifies naturally with the set of partitions of X into two subsets, and will be denoted $P_2(X)$. If $p: P_2(X) \to \mathbb{Z}/2\mathbb{Z}$ still denotes the induced map, we will write

$$P_{2}^{+}(X) = p^{-1}(0)$$
$$P_{2}^{-}(X) = p^{-1}(1).$$

With respect to the bilinear form e, X is orthogonal to 2_+^X , then inducing an alternate bilinear form, still denoted by e, on $P_2^+(X)$. This form is *nondegenerate*. To prove this, observe that if $A \in 2_+^X$, A different from \emptyset and X, and $x \in A$, $x' \notin A$; then, if $A' = \{x, x'\}$, e(A, A') = 1.

3.6 Two cases may appear in this situation.

a) g is even. Then, the map $Q: 2_{-}^{X} \to \mathbb{Z}/2\mathbb{Z}$ passes to the quotient $P_{2}^{-}(X)$, so this becomes a symplectic torsor over $(P_{2}^{+}(X), e)$. But in this case the canonical quadratic form q_{o} does not pass to the quotient $P_{2}^{+}(X)$.

b) g is odd. Then, the map Q does not pass to the quotient, but q_o does, so there is a natural characteristic.

3.7 The following construction would help in developing the case where |X| is even along the lines of 3.2-3.5, which I won't do. Let X be of odd order |X| = 2g + 1, and define $X' = X \coprod \{X\}$, thus |X'| = 2g + 2. We have a natural linear map

$$2^X \rightarrow 2^{X'}$$

and this is compatible with p, e, Q, q_o . Composing this with the passage to the quotient, I have a linear isomorphism

$$2^X \to P_2(X'),$$

and by compatibility with p, p', isomorphisms

$$2^{X}_{+} \to P^{+}_{2}(X')$$
$$2^{X}_{-} \to P^{-}_{2}(X').$$

The first is compatible with e, e', and with the canonical quadratic forms if g is odd. The second is compatible with Q, Q' if g is even.

$\S 4$ Basis and fundamental sets

4.1 Normal basis. Let (J, e) be a symplectic pair. A normal basis for (J, e) is a basis $(x_i)_{i \in I}$ for J with the property that $e(x_i, x_j) = 1$ for $i \neq j$, the set of ordered normal basis (i.e. for $I = \{1, ..., 2g\}$ if $2g = \dim J$) will be denoted ONB(J, e). The symplectic group Sp (J, e) clearly acts on ONB(J, e) and it does it simply transitively, because if two ordered normal bases for (J, e) are given, the unique linear automorphism transforming one into the other is obviously symplectic.

I have not yet shown that the set ONB(J, e) is non-empty, this we will see as a consequence of the following construction, that relates symplectic basis (0.1) with normal basis. The set SB(J, e) of symplectic basis is a torsor over Sp(J, e), thus if ONB(J, e) is non-empty, both torsors should be isomorphic and indeed there would be as many isomorphisms as elements in the group Sp(J, e). What I proceed to exhibit now is a definite isomorphism

$$\alpha \colon SB(J, e) \to ONB(J, e)$$

with inverse β . If

$$x \in SB(J, e), x = (x_1, ..., x_g, x'_1, ..., x'_g)$$

let's put $y = \alpha(x)$, then by definition