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2.7 COROLLARY. Let $(S, Q), (S', Q')$ be symplectic torsors of genus g over $(J, e), (J', e')$, and let $\Sigma = S^\pm, \Sigma' = S'^\pm$. Then, there are canonical bijections

$$\begin{aligned} \text{Isom}((J, e), (J', e')) &\simeq \text{Isom}((S, Q), (S', Q')) \\ &\simeq \text{Isom}((\Sigma, \Sigma_{(4)}), (\Sigma', \Sigma'_{(4)})). \end{aligned}$$

In particular, there are group isomorphisms

$$\text{Sp}(J, e) \simeq \text{Sp}(S, Q) \simeq \text{Aut}(\Sigma, \Sigma_{(4)}).$$

§ 3 SYMPLECTIC TORSORS DEFINED BY FINITE SETS

In this paragraph, X will be a finite set.

3.1 The basic construction. Starting from X one has

a) The set 2^X of subsets of X , with the operation of symmetric difference:

$$A + B = A \cup B - A \cap B \quad A, B \in 2^X$$

b) A map $p: 2^X \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by

$$p(A) = |A| \pmod{2} \quad A \in 2^X$$

c) A map $e: 2^X \times 2^X \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by

$$e(A, B) = |A \cap B| \pmod{2} \quad A, B \in 2^X$$

d) A map $Q: 2_-^X \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by

$$Q(B) = \frac{|B| + 1}{2} \pmod{2} \quad B \in 2_-^X$$

where $2_-^X = p^{-1}(1)$ is the set of subsets of odd order of X .

e) A map $q_0: 2_+^X \rightarrow \mathbb{Z}/2\mathbb{Z}$ defined by

$$q_0(A) = \frac{|A|}{2} \pmod{2} \quad A \in 2_+^X$$

where $2_+^X = p^{-1}(0)$.

Then, it is easily verified that

$\alpha)$ 2^X is a vector space over $\mathbb{Z}/2\mathbb{Z}$, of dimension $|X|$.

$\beta)$ p is linear

$\gamma)$ e is bilinear

δ) Q has the following property (compare 1.1.1)

$$Q(B) + Q(A+B) + Q(A'+B) + Q(A+A'+B) = e(A, A')$$

whenever $B \in 2_-^X$, $A, A' \in 2_+^X$

ε) q_o is a quadratic form inducing the restriction of e to 2_+^X .

In the proof of these, one uses the following identity

$$|A + B| = |A| + |B| - 2|A \cap B| \quad A, B \in 2^X.$$

3.2 Let's assume in the following three sections that X is of odd order, $|X| = 2g + 1$.

3.2.1 PROPOSITION. *The bilinear form e on 2_+^X is alternate and non-degenerate. If 2_+^X acts on 2_-^X by translations, $(2_-^X, Q)$ is a symplectic torsor over $(2_+^X, e)$ which is even for $g \equiv 2, 3 \pmod{4}$ and odd for $g \equiv 0, 1 \pmod{4}$.*

3.2.2 Proof. It is clear that e is alternate on 2_+^X . It is also non degenerate, because if $A \in 2_+^X$, $A \neq \phi$, let $x \in A$; then $A' = (X - A) \cup \{x\}$ is of even order, and $e(A, A') = 1$. It is also clear that $(2_-^X, Q)$ is a symplectic torsor over $(2_+^X, e)$ (because of 3.1 δ) and the definition of symplectic torsor.

To find out when this torsor is even or odd, we first observe that it is clearly odd for $g = 0, 1$ (look at it), then apply descending induction using the following fact (to be proved below). Let's call ε_g the type of the torsor corresponding to an X with $|X| = 2g + 1$ (and $g \geq 2$), thus $\varepsilon_g = \pm 1$; then $\varepsilon_g = \varepsilon_{g-1}$ if g is odd, and $\varepsilon_g = -\varepsilon_{g-1}$ if g is even.

Proof of this fact: take a fixed $A_o \subset X$ of order two. The set of $B \in 2_-^X$ such that $Q(B) = Q(A_o + B) = 0$ (recall that $Q(B) = 0$ means that $|B| \equiv 1 \pmod{4}$) has cardinality $2^{g-1} (2^{g-1} + \varepsilon_g)$ by definition of ε_g and proposition 2.1.1. But clearly this number is also twice the cardinality of the set of subsets C of $X - A_o$ such that $|C| \equiv 2g - 1 \pmod{4}$ (in fact any such B defines a C by $C = X - (A_o \cup B)$ and this map is two-fold) and the number of these is $2^{g-2} (2^{g-1} + \varepsilon_{g-1})$ or $2^{g-2} (2^{g-1} - \varepsilon_{g-1})$ according to $2g - 1 \equiv 1 \pmod{4}$ or $2g - 1 \equiv 3 \pmod{4}$, i.e. g odd or even. This proves the fact and completes the proof of the proposition.

3.3 If Q is odd, let us agree to modify Q in the way described in 1.1 to obtain an even torsor \bar{Q} . With this convention, the following notation will be adopted:

$$\begin{aligned} J_X &= 2_+^X & e_X &= e \\ S_X &= 2_-^X & Q_X &= Q \end{aligned}$$

or \bar{Q} according to the value of $g \pmod{4}$.

The identification $S_X \simeq Q(J_X, e_X)$ in 1.4 may be made explicit: if $B \in S_X$, B becomes the following quadratic form

$$B(A) = |A \cap B| + \frac{|A|}{2}(2).$$

Let's now make explicit the condition for a triple (B_1, B_2, B_3) of elements of either S_X^+ or S_X^- to be a *triplet* (2.3). This means that

$$Q_X(\Sigma B_i) = \Sigma Q_X(B_i),$$

and this is equivalent to

$$\sum_{i < j} |B_i \cap B_j| \equiv 1(2),$$

or still to

$$|\cup B| \equiv |\cap B_i|(2).$$

3.4 The quadratic form q_o on J_X singled out in 3.1 e) corresponds through the identification $Q(J_X, e_X) = S_X$ to X itself. As $Q(X) \equiv g + 1(2)$, it results from the last part of 3.2.1 that the Arf invariant of q_o is 0 for $g \equiv 0, 3(4)$, 1 for $g \equiv 1, 2(4)$. In other words, $q_o \in S_X^+$ for $g \equiv 0, 3(4)$, $q_o \in S_X^-$ for $g \equiv 1, 2(4)$.

3.5 Let's assume in this and the next sections that X is of even order, $|X| = 2g + 2$. Then, the linear map p passes to the quotient $2^X / \{0, X\}$. This quotient identifies naturally with the set of partitions of X into two subsets, and will be denoted $P_2(X)$. If $p: P_2(X) \rightarrow \mathbf{Z}/2\mathbf{Z}$ still denotes the induced map, we will write

$$P_2^+(X) = p^{-1}(0)$$

$$P_2^-(X) = p^{-1}(1).$$

With respect to the bilinear form e , X is orthogonal to 2_+^X , then inducing an alternate bilinear form, still denoted by e , on $P_2^+(X)$. This form is *non-degenerate*. To prove this, observe that if $A \in 2_+^X$, A different from \emptyset and X , and $x \in A$, $x' \notin A$; then, if $A' = \{x, x'\}$, $e(A, A') = 1$.

3.6 Two cases may appear in this situation.

a) g is even. Then, the map $Q: 2_-^X \rightarrow \mathbf{Z}/2\mathbf{Z}$ passes to the quotient $P_2^-(X)$, so this becomes a symplectic torsor over $(P_2^-(X), e)$. But in this case the canonical quadratic form q_o does not pass to the quotient $P_2^+(X)$.

b) g is odd. Then, the map Q does not pass to the quotient, but q_o does, so there is a natural characteristic.

3.7 The following construction would help in developing the case where $|X|$ is even along the lines of 3.2-3.5, which I won't do. Let X be of odd order $|X| = 2g + 1$, and define $X' = X \amalg \{X\}$, thus $|X'| = 2g + 2$. We have a natural linear map

$$2^X \rightarrow 2^{X'}$$

and this is compatible with p, e, Q, q_0 . Composing this with the passage to the quotient, I have a linear isomorphism

$$2^X \rightarrow P_2(X'),$$

and by compatibility with p, p' , isomorphisms

$$\begin{aligned} 2_+^X &\rightarrow P_2^+(X') \\ 2_-^X &\rightarrow P_2^-(X'). \end{aligned}$$

The first is compatible with e, e' , and with the canonical quadratic forms if g is odd. The second is compatible with Q, Q' if g is even.

§ 4 BASIS AND FUNDAMENTAL SETS

4.1 *Normal basis.* Let (J, e) be a symplectic pair. A *normal basis* for (J, e) is a basis $(x_i)_{i \in I}$ for J with the property that $e(x_i, x_j) = 1$ for $i \neq j$, the set of ordered normal basis (i.e. for $I = \{1, \dots, 2g\}$ if $2g = \dim J$) will be denoted $ONB(J, e)$. The symplectic group $Sp(J, e)$ clearly acts on $ONB(J, e)$ and it does it simply transitively, because if two ordered normal bases for (J, e) are given, the unique linear automorphism transforming one into the other is obviously symplectic.

I have not yet shown that the set $ONB(J, e)$ is non-empty, this we will see as a consequence of the following construction, that relates symplectic basis (0.1) with normal basis. The set $SB(J, e)$ of symplectic basis is a torsor over $Sp(J, e)$, thus if $ONB(J, e)$ is non-empty, both torsors should be isomorphic and indeed there would be as many isomorphisms as elements in the group $Sp(J, e)$. What I proceed to exhibit now is a definite isomorphism

$$\alpha: SB(J, e) \rightarrow ONB(J, e)$$

with inverse β . If

$$x \in SB(J, e), x = (x_1, \dots, x_g, x'_1, \dots, x'_g)$$

let's put $y = \alpha(x)$, then by definition