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## HOW TO USE RUNGE'S THEOREM

by L. A. RUBEL

Runge's Theorem on approximation to analytic functions by polynomials is a powerful tool, and belongs in every analyst's bag of tricks. We illustrate how it can be used by giving three applications of it here. The first two answer problems put to us by D. J. Newman, although we don't believe they originated with him. The third concerns a problem that seems to be part of the folklore. Rather than attempt a detailed history, we merely cite [1], [2], [3, p. 221], [4], [5, §4], and [7]. We give fully detailed proofs, although there are simple geometrical ideas that underlie them. It is our hope that this expository paper will help equip the reader with Runge's Theorem as a versatile tool.

**RUNGE'S THEOREM** [1, p. 177]. If  $G$  is an open set in the complex plane whose complement on the Riemann sphere is connected, if  $f$  is an analytic function on  $G$ , if  $K$  is a compact subset of  $G$  and if  $\varepsilon > 0$ , then there exists a polynomial  $P$  such that  $|P(z) - f(z)| < \varepsilon$  for all  $z \in K$ .

**PROBLEM 1.** Let  $H$  be the class of real-valued functions that are harmonic in the open unit disc  $D = \{z: |z| < 1\}$  and let  $H_0$  consist of those functions  $u \in H$  for which  $u(0) = 0$ . Let  $N_u = \{z \in D: u(z) \leq 0\}$  and  $m(u)$  be the area of  $N_u$ . Does there exist an  $\varepsilon > 0$  such that  $m(u) \geq \varepsilon$  for all  $u \in H_0$ ?

*Solution.* It is plausible to some people that such an  $\varepsilon$  exists, since by the Gauss mean value theorem,  $\iint_{D_r} u \, dx \, dy = 0$  for  $r < 1$ , where  $D_r = \{z: |z| < r\}$ , so that in one sense, the negative values of  $u$  balance out the positive values. Nevertheless, there is no such  $\varepsilon$ .

*Proof.* Take  $\delta$  with  $0 < \delta < 1/2$  and consider the open set  $G_\delta = D_{\delta/2} \cup J_\delta$ , where  $J_\delta$  is the keyhole region

$$J_\delta = D_{1-\delta} \setminus (D_\delta^- \cup A_\delta),$$

where

$$A_\delta = \{z = re^{i\theta}: -\delta \leq \theta \leq \delta\}$$

and  $V^-$  denotes the topological closure of  $V$ . Let  $f$  be defined in  $G_\delta$  by  $f(z) = 10$  for  $z \in J_\delta$  and  $f(z) = 0$  for  $z \in D_{\delta/2}$ . Notice that  $f$  is analytic on  $G_\delta$  and that  $G_\delta$  has a connected complement. Let

$$K_\delta = D_{\delta/4}^- \cup J_{2\delta}^-,$$

so that  $K_\delta$  is a compact subset of  $G_\delta$ . By Runge's theorem, there is a polynomial  $P = P_\delta$  such that

$$\sup \{|P(z) - f(z)| : z \in K_\delta\} < 1.$$

Let

$$u(z) = \operatorname{Re}(P(z) - P(0))$$

so that  $u(0) = 0$  and  $u(z) \geq 8$  for  $z \in J_{2\delta}^-$ . Hence  $u = u_\delta \in H_0$  and  $N_u \subseteq D \setminus J_{2\delta}^-$ . But it is easy to verify that the limit of the area of  $J_{2\delta}^-$  as  $\delta \rightarrow 0$  is the area,  $\pi$ , of  $D$ , so that  $\lim_{\delta \rightarrow 0} m(u_\delta) = 0$ , and there cannot therefore exist an  $\varepsilon > 0$  with the desired property.

**PROBLEM 2.** For  $u \in H_0$ , let  $L$  be the level line of  $u$  that passes through 0, i.e.  $L$  is that component of  $\{z : u(z) = 0\}$  that passes through 0. Let

$$L_{1/2} = \{z \in L : |z| \geq 1/2\}.$$

Is there a finite number  $M$  such that if  $u$  is not identically 0, then the length of  $L_{1/2}$  does not exceed  $M$ ?

*Solution.* The answer is no. We force  $L$  to wiggle past so many suitably placed barriers that the length of  $L_{1/2}$  must be large.

*Proof.* Choose a positive integer  $n$ , and let

$$K_n = \{0\} \cup \left( \bigcup_{j=1}^{n-1} A_{2j}^n \right) \cup \left( \bigcup_{j=1}^n B_{2j-1}^n \right)$$

where for  $v = 1, 2, \dots, 2n-1$ ,

$$A_v^n = \{z = re^{i\theta} : r = v/2n, -\pi/10 \leq \theta \leq \pi/10\}$$

$$B_v^n = \{z = re^{i\theta} : r = v/2n, -\pi + \pi/10 \leq \theta \leq \pi - \pi/10\}.$$

Further, let

$$G_v^n = \left\{ z = re^{i\theta} : \frac{v}{2n} - \frac{1}{10n} < r < \frac{v}{2n} + \frac{1}{10n}, -\frac{\pi}{20} < \theta < \frac{\pi}{20} \right\}$$

$$H_v^n = \left\{ z = re^{i\theta} : \frac{v}{2n} - \frac{1}{10n} < r < \frac{v}{2n} + \frac{1}{10n}, -\pi + \frac{\pi}{20} < \theta < \pi - \frac{\pi}{20} \right\},$$

and let

$$\Omega_n = D_{n/20} \cup \left( \bigcup_{j=1}^{n-1} G_{2j}^n \right) \cup \left( \bigcup_{j=1}^n H_{2j-1}^n \right).$$

Clearly,  $\Omega_n$  is an open set in  $D$  whose complement is connected, and each component of  $K_n$  lies in exactly one component of  $\Omega_n$ . Let  $f_n$  be defined by  $f_n(z) = 0$  for  $z \in D_{n/20}$ , and  $f_n(z) = 10$  for  $z$  in the other components of  $\Omega_n$ , so that  $f_n$  is an analytic function on  $\Omega_n$ , and we may apply Runge's theorem to find a polynomial  $P = P_n$  so that

$$\sup \{ |P_n(z) - f_n(z)| : z \in K_n \} < 1.$$

In particular,  $P_n(0) < 1$  and  $P_n(z) > 9$  for  $z \in \Gamma_n = K_n \setminus \{0\}$ . Let

$$u_n(z) = \operatorname{Re} \{ P_n(z) - P_n(0) \}$$

so that  $u_n \in H_0$  and  $u_n(z) \geq 8$  for  $z \in \Gamma_n$ . Now by the maximum modulus theorem, the level line  $L^n$  of  $u_n$  through 0 must extend to the boundary of  $D$ . Yet it must avoid  $\Gamma_n$ . It is then easy to see that there is a positive constant  $c$  such that the length of  $L_{1/2}^n$  exceeds  $cn$ , so that there is no upper bound on the length of  $L_{1/2}$ .

**PROBLEM 3.** Do there exist two analytic functions  $f_1$  and  $f_2$  in  $D$  such that

$$\liminf_{r \rightarrow 1} \{ |f_1(z)| + |f_2(z)| : |z| \geq r \} = \infty ?$$

*Solution.* We construct such a pair  $f_1, f_2$  below, by using a gap series to define  $f_1$  and then Runge's theorem to define  $f_2$ . It is easy to see from the minimum modulus theorem that there does not exist a single analytic function  $f$  in  $D$  such that

$$\liminf_{r \rightarrow 1} \{ |f(z)| : |z| \geq r \} = \infty.$$

*Proof.* First, we construct  $f_1$ . Choose  $n_1 = 5$ , and then choose  $r_1$  with  $0 < r_1 < 1$  so that  $n_1 r_1^{n_1-1} > 3$ . Then choose a positive integer  $m_1$  so that

$$n_1 r_1^{n_1-1} \geq 3 + \sum_{j=m_1}^{\infty} j r_1^j.$$

Having constructed  $n_1, \dots, n_k; r_1, \dots, r_k; m_1, \dots, m_k$ , proceed as follows. Choose a positive integer  $n_{k+1} > m_k$  so that

$$n_{k+1} > k + 3 + \sum_{j=1}^k n_j.$$

Now choose  $r_{k+1}$ , with  $r_k < r_{k+1} < 1$ , so that

$$n_{k+1} r_{k+1}^{n_{k+1}} > k + 1 + \sum_{j=1}^k n_j.$$

Hence

$$n_{k+1} r_{k+1}^{n_{k+1}} > k + 1 + \sum_{j=1}^k n_j r_{k+1}^j.$$

Then choose a positive integer  $m_{k+1}$  so that

$$n_{k+1} r_{k+1}^{n_{k+1}} \geq k + 1 + \sum_{j=1}^k n_j r_{k+1}^j + \sum_{j=m_{k+1}}^{\infty} j r_{k+1}^j.$$

Let

$$f_1(z) = \sum_{k=1}^{\infty} n_k z^{n_k}.$$

It is clear that

$$\lim_{x \rightarrow 1-} f_1(x) = +\infty.$$

We claim that  $|f_1(z)| \geq k + 1$  for  $|z| = r_{k+1}$ . To see this, we write

$$\begin{aligned} |f(r_{k+1}e^{i\theta})| &\geq n_{k+1} r_{k+1}^{n_{k+1}} - \sum_{j=1}^k n_j r_{k+1}^{n_j} - \sum_{j=k+1}^{\infty} n_j r_{k+1}^{n_j} \\ &\geq n_{k+1} r_{k+1}^{n_{k+1}} - \sum_{j=1}^k n_j r_{k+1}^{n_j} - \sum_{j=m_{k+1}}^{\infty} j r_{k+1}^j \geq k + 1. \end{aligned}$$

So

$$\liminf_{r \rightarrow 1} \{|f_1(z)| : z \in E, |z| \geq r\} = \infty,$$

where

$$E = [0, 1) \cup \left( \bigcup_{k=1}^{\infty} \{z : |z| = r_k\} \right).$$

By continuity

$$(*) \quad \liminf_{r \rightarrow 1} \{|f_1(z)| : z \in G, |z| \geq r\} = \infty$$

where  $G$  is some open superset in  $D$  of the set  $E$ . Now to construct  $f_2$ , we observe that the complement in  $D$  of  $G$  is contained in  $\bigcup_{k=1}^{\infty} S_k$ , where each  $S_k$  is a closed annular sector of the form

$$S_k = \{z = re^{i\theta} : s_k \leq r \leq t_k, \delta_k \leq \theta \leq 2\pi - \delta_k\}$$

and  $s_k \uparrow 1$ ,  $t_k \uparrow 1$  and  $t_k < s_{k+1}$  for  $k = 1, 2, 3, \dots$ .

Define  $g_1$  in  $D$  by  $g_1(z) = 2$ . Having defined  $g_1, \dots, g_n$ , consider a closed disc  $U^n$  with center at 0 that contains  $S_1, \dots, S_n$  and a slightly larger open disc  $D^n$  that excludes  $S_{n+1}$ . Let  $S'_{n+1}$  be an open superset of  $S_{n+1}$  that does not intersect  $D^n$ , and define  $\varphi_n$  in  $D^n \cup S'_{n+1}$  by  $\varphi_n(z) = n + 2 - \sum_{i=1}^n g_i(z)$  for  $z \in S'_{n+1}$  and  $\varphi_n(z) = 0$  in  $D^n$ . By Runge's theorem, there is a polynomial  $g_n$  such that

$$|g_n(z) - \varphi_n(z)| < 2^{-n-2}$$

for  $z \in U^n \cup S_{n+1}$ . Let  $f_2 = \sum_{j=1}^{\infty} g_j$ . It is easily verified that the series converges uniformly on compact subsets of  $D$  to a function  $f_2$  that is analytic on  $D$ . On  $S_n$ ,

$$f_2(z) = g_n(z) + \sum_{i=1}^{n-1} g_i(z) + \sum_{i=n+1}^{\infty} g_i(z),$$

so that in  $S_n$

$$|f_2(z)| \geq n + 1 - \sum_{i=n+1}^{\infty} 2^{-i-2} \geq n.$$

Hence

$$(**) \quad \liminf_{r \rightarrow 1} \{ |f_2(z)| : z \in \bigcup_{n=1}^{\infty} S_n, |z| \geq r \} = \infty$$

Since  $G \cup \left( \bigcup_{n=1}^{\infty} S_n \right) = D$ , we have the desired result on putting (\*) and (\*\*) together.

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