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HOW TO USE RUNGE'S THEOREM

by L. A. RUBEL

Runge's Theorem on approximation to analytic functions by polynomials is a powerful tool, and belongs in every analyst's bag of tricks. We illustrate how it can be used by giving three applications of it here. The first two answer problems put to us by D. J. Newman, although we don't believe they originated with him. The third concerns a problem that seems to be part of the folklore. Rather than attempt a detailed history, we merely cite [1], [2], [3, p. 221], [4], [5, §4], and [7]. We give fully detailed proofs, although there are simple geometrical ideas that underlie them. It is our hope that this expository paper will help equip the reader with Runge's Theorem as a versatile tool.

RUNGE'S THEOREM [1, p. 177]. If G is an open set in the complex plane whose complement on the Riemann sphere is connected, if f is an analytic function on G, if K is a compact subset of G and if $\varepsilon > 0$, then there exists a polynomial P such that $|P(z) - f(z)| < \varepsilon$ for all $z \in K$.

PROBLEM 1. Let H be the class of real-valued functions that are harmonic in the open unit disc $D = \{z : |z| < 1\}$ and let H_0 consist of those functions $u \in H$ for which $u(0) = 0$. Let $N_u = \{z \in D : u(z) \leqslant 0\}$ and $m(u)$ be the area of N_u. Does there exist an $\varepsilon > 0$ such that $m(u) \geqslant \varepsilon$ for all $u \in H_0$?

Solution. It is plausible to some people that such an ε exists, since by the Gauss mean value theorem, $\iint_{D_r} u \, dx \, dy = 0$ for $r < 1$, where $D_r = \{z : |z| < r\}$, so that in one sense, the negative values of u balance out the positive values. Nevertheless, there is no such ε.

Proof. Take δ with $0 < \delta < 1/2$ and consider the open set $G_\delta = D_{\delta/2} \cup J_\delta$, where J_δ is the keyhole region

$$J_\delta = D_{1-\delta} \backslash (D_\delta^- \cup A_\delta) ,$$

where

$$A_\delta = \{z = re^{i\theta} : -\delta \leqslant \theta \leqslant \delta\}$$

and V^- denotes the topological closure of V. Let f be defined in G_δ by $f(z) = 10$ for $z \in J_\delta$ and $f(z) = 0$ for $z \in D_{\delta/2}$. Notice that f is analytic on G_δ and that G_δ has a connected complement. Let

$$K_\delta = D^-_{\delta/4} \cup J^-_{2\delta},$$

so that K_δ is a compact subset of G_δ. By Runge's theorem, there is a polynomial $P = P_\delta$ such that

$$sup \{|P(z) - f(z)|: z \in K_\delta\} < 1 .$$

Let

$$u(z) = Re\,(P(z) - P(0))$$

so that $u(0) = 0$ and $u(z) \geqslant 8$ for $z \in J^-_{2\delta}$. Hence $u = u_\delta \in H_0$ and $N_u \subseteq D \backslash J^-_{2\delta}$. But it is easy to verify that the limit of the area of $J^-_{2\delta}$ as $\delta \to 0$ is the area, π, of D, so that $\lim_{\delta -> 0} m(u_\delta) = 0$, and there cannot therefore exist an $\varepsilon > 0$ with the desired property.

Problem 2. For $u \in H_0$, let L be the level line of u that passes through 0, i.e. L is that component of $\{z: u(z) = 0\}$ that passes through 0. Let

$$L_{1/2} = \{z \in L: |z| \geqslant 1/2\} .$$

Is there a finite number M such that if u is not identically 0, then the length of $L_{1/2}$ does not exceed M?

Solution. The answer is no. We force L to wiggle past so many suitably placed barriers that the length of $L_{1/2}$ must be large.

Proof. Choose a positive integer n, and let

$$K_n = \{0\} \cup \left(\bigcup_{j=1}^{n-1} A^n_{2j}\right) \cup \left(\bigcup_{j=1}^{n} B^n_{2j-1}\right)$$

where for $v = 1, 2, ..., 2n-1$,

$$A^n_v = \{z = re^{i\theta}: r = v/2n, \ -\pi/10 \leqslant \theta \leqslant \pi/10\}$$

$$B^n_v = \{z = re^{i\theta}: r = v/2n, \ -\pi + \pi/10 \leqslant \theta \leqslant \pi - \pi/10\} .$$

Further, let

$$G^n_v = \left\{z = re^{i\theta}: \frac{v}{2n} - \frac{1}{10n} < r < \frac{v}{2n} + \frac{1}{10n}, \ -\frac{\pi}{20} < \theta < \frac{\pi}{20}\right\}$$

$$H_\nu^n = \left\{z = re^{i\theta}: \frac{\nu}{2n} - \frac{1}{10n} < r < \frac{\nu}{2n} + \frac{1}{10n}, -\pi + \frac{\pi}{20} < \theta < \pi - \frac{\pi}{20}\right\},$$

and let

$$\Omega_n = D_{n/20} \cup \left(\bigcup_{j=1}^{n-1} G_{2j}^n\right) \cup \left(\bigcup_{j=1}^{n} H_{2j-1}^n\right).$$

Clearly, Ω_n is an open set in D whose complement is connected, and each component of K_n lies in exactly one component of Ω_n. Let f_n be defined by $f_n(z) = 0$ for $z \in D_{n/20}$, and $f_n(z) = 10$ for z in the other components of Ω_n, so that f_n is an analytic function on Ω_n, and we may apply Runge's theorem to find a polynomial $P = P_n$ so that

$$sup \{|P_n(z) - f_n(z)|: z \in K_n\} < 1.$$

In particular, $P_n(0) < 1$ and $P_n(z) > 9$ for $z \in \Gamma_n = K_n \backslash \{0\}$. Let

$$u_n(z) = Re \{P_n(z) - P_n(0)\}$$

so that $u_n \in H_0$ and $u_n(z) \geqslant 8$ for $z \in \Gamma_n$. Now by the maximum modulus theorem, the level line L^n of u_n through 0 must extend to the boundary of D. Yet it must avoid Γ_n. It is then easy to see that there is a positive constant c such that the length of $L_{1/2}^n$ exceeds cn, so that there is no upper bound on the length of $L_{1/2}$.

PROBLEM 3. Do there exist two analytic functions f_1 and f_2 in D such that

$$\liminf_{r \to 1} \{|f_1(z)| + |f_2(z)|: |z| \geqslant r\} = \infty ?$$

Solution. We construct such a pair f_1, f_2 below, by using a gap series to define f_1 and then Runge's theorem to define f_2. It is easy to see from the minimum modulus theorem that there does not exist a single analytic function f in D such that

$$\liminf_{r \to 1} \{|f(z)| : |z| \geqslant r\} = \infty .$$

Proof. First, we construct f_1. Choose $n_1 = 5$, and then choose r_1 with $0 < r_1 < 1$ so that $n_1 r_1^{n_1} > 3$. Then choose a positive integer m_1 so that

$$n_1 r_1^{n_1} \geqslant 3 + \sum_{j=m_1}^{\infty} j r^j .$$

Having constructed $n_1, \ldots, n_k$; $r_1, \ldots, r_k$; $m_1, \ldots, m_k$, proceed as follows. Choose a positive integer $n_{k+1} > m_k$ so that

$$n_{k+1} > k + 3 + \sum_{j=1}^{k} n_j .$$

Now choose r_{k+1}, with $r_k < r_{k+1} < 1$, so that

$$n_{k+1} \, r_{k+1}^{n_{k+1}} > k + 1 + \sum_{j=1}^{k} n_j .$$

Hence

$$n_{k+1} \, r_{k+1}^{n_{k+1}} > k + 1 + \sum_{j=1}^{k} n_j \, r_{k+1}^j .$$

Then choose a positive integer m_{k+1} so that

$$n_{k+1} \, r_{k+1}^{n_{k+1}} \geqslant k + 1 + \sum_{j=1}^{k} n_j \, r_{k+1}^{n_j} + \sum_{j=m_{k+1}}^{\infty} j \, r_{k+1}^j .$$

Let

$$f_1(z) = \sum_{k=1}^{\infty} n_k \, z^{n_k} .$$

It is clear that

$$\lim_{x \to 1-} f_1(x) = +\infty .$$

We claim that $|f_1(z)| \geqslant k + 1$ for $|z| = r_{k+1}$. To see this, we write

$$\begin{aligned} |f(r_{k+1}e^{i\theta})| &\geqslant n_{k+1} \, r_{k+1}^{n_{k+1}} - \sum_{j=1}^{k} n_j \, r_{k+1}^{n_j} - \sum_{j=k+1}^{\infty} n_j \, r_{k+1}^{n_j} \\ &\geqslant n_{k+1} \, r_{k+1}^{n_{k+1}} - \sum_{j=1}^{k} n_j \, r_{k+1}^{n_j} - \sum_{j=m_{k+1}}^{\infty} j \, r_{k+1}^j \geqslant k + 1 . \end{aligned}$$

So

$$\liminf_{r \to 1} \{ |f_1(z)| : z \in E, |z| \geqslant r \} = \infty ,$$

where

$$E = [0, 1) \cup \left(\bigcup_{k=1}^{\infty} \{ z : |z| = r_k \} \right) .$$

By continuity

(*) $$\liminf_{r \to 1} \{ |f_1(z)| : z \in G, |z| \geqslant r \} = \infty$$

where G is some open superset in D of the set E. Now to construct f_2, we observe that the complement in D of G is contained in $\bigcup_{k=1}^{\infty} S_k$, where each S_k is a closed annular sector of the form

$$S_k = \{z = re^{i\theta} : s_k \leqslant r \leqslant t_k, \delta_k \leqslant \theta \leqslant 2\pi - \delta_k\}$$

and $s_k \uparrow 1$, $t_k \uparrow 1$ and $t_k < s_{k+1}$ for $k = 1, 2, 3, \ldots$.

Define g_1 in D by $g_1(z) = 2$. Having defined $g_1, \ldots, g_n$, consider a closed disc U^n with center at 0 that contains $S_1, \ldots, S_n$ and a slightly larger open disc D^n that excludes S_{n+1}. Let S'_{n+1} be an open superset of S_{n+1} that does not intersect D^n, and define φ_n in $D^n \cup S'_{n+1}$ by $\varphi_n(z) = n + 2 - \sum_{i=1}^{n} g_1(z)$ for $z \in S'_{n+1}$ and $\varphi_n(z) = 0$ in D^n. By Runge's theorem, there is a polynomial g_n such that

$$|g_n(z) - \varphi_n(z)| < 2^{-n-2}$$

for $z \in U^n \cup S_{n+1}$. Let $f_2 = \sum_{j=1}^{\infty} g_j$. It is easily verified that the series converges uniformly on compact subsets of D to a function f_2 that is analytic on D. On S_n,

$$f_2(z) = g_n(z) + \sum_{i=1}^{n-1} g_i(z) + \sum_{i=n+1}^{\infty} g_i(z),$$

so that in S_n

$$|f_2(z)| \geqslant n + 1 - \sum_{i=n+1}^{\infty} 2^{-i-2} \geqslant n .$$

Hence

$$(**) \qquad \liminf_{r \to 1} \{|f_2(z)| : z \in \bigcup_{n=1}^{\infty} S_n, |z| \geqslant r\} = \infty$$

Since $G \cup \left(\bigcup_{n=1}^{\infty} S_n\right) = D$, we have the desired result on putting (*) and (**) together.

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