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(3) E has non-split multiplicative reduction at $2 < = > a_1 \equiv 1 \pmod{2}$ and $a_2 + a_3 \equiv 1 \pmod{2}$.

Proof: (1). $c_4 \equiv b_2^2 - 24b_4 \equiv b_2^2 \equiv b_2 \equiv a_1^2 + 4a_2 \equiv a_1^2 \equiv a_1 \equiv C_1 \pmod{2}$. Now apply Corollary 1.2, part (2).

(2) and (3). By Corollary 1.2, part (2), we have multiplicative reduction $< = > a_1 \equiv 1 \pmod{2}$. Assume that this is so. Let $S = (x, y)$ be the singular point. Let

$$(2.3) \quad H = Y^2 + a_1XY + a_3Y - X^3 - a_2X^2 - a_4X - a_6$$

Compute in $\mathbf{Z}/2\mathbf{Z}$ for the remainder of the proof.

$$(2.4) \quad \frac{\partial H}{\partial X} = a_1Y - 3X^2 - 2a_2X - a_4 = Y + X^2 + a_4$$

$$(2.5) \quad \frac{\partial H}{\partial Y} = 2Y + a_1X + a_3 = X + a_3$$

$x = a_3$ from (2.5) and $y = x^2 + a_4 = x + a_4 = a_3 + a_4$ from (2.4). Transform S to $(0, 0)$ via $X \rightarrow X + a_3$ and $Y \rightarrow Y + a_3 + a_4$. We obtain

$$\begin{aligned} H &= (Y + a_3 + a_4)^2 + a_1(X + a_3)(Y + a_3 + a_4) + a_3(Y + a_3 + a_4) \\ &\quad - (X + a_3)^3 - a_2(X + a_3)^2 - a_4(X + a_3) - a_6 \\ &= Y^2 + XY + X^3 + (a_2 + a_3)X^2 \end{aligned}$$

The tangents at $(0, 0)$ are given by $Y^2 + XY + (a_2 + a_3)X^2 = 0$. E has split multiplicative reduction at $2 < = >$ this form is reducible over $\mathbf{Z}/2\mathbf{Z} < = > a_2 + a_3 \equiv 0 \pmod{2}$.

§3. THE CASE $p = 3$

As in §2, a short computation (again see Tate [5] for the details) yields

$$(3.1) \quad C_2 = a_1^2 + a_2$$

THEOREM 3.1. Assume E has bad reduction at 3.

- (1) E has additive reduction at 3 $\Leftrightarrow a_1^2 + a_2 \equiv 0 \pmod{3}$. $\Leftrightarrow c_4 \equiv 0 \pmod{3}$.
- (2) E has multiplicative reduction at 3 $\Leftrightarrow a_1^2 + a_2 \not\equiv 0 \pmod{3}$ $\Leftrightarrow c_4 \not\equiv 0 \pmod{3}$.

(3) E has split multiplicative reduction at 3 $\Leftrightarrow a_1^2 + a_2 \equiv 1 \pmod{3}$.

(4) E has non-split multiplicative reduction at 3 $\Leftrightarrow a_1^2 + a_2 \equiv -1 \pmod{3}$.

Proof:

$$c_4 \equiv b_2^2 - 24b_4 \equiv b_2^2 \equiv (a_1^2 + 4a_2)^2 \equiv (a_1^2 + a_2)^2 \pmod{3}.$$

The theorem then follows immediately from formula (3.1) and Corollary 1.2.

Remark. $C_2^2 \equiv c_4 \pmod{3}$. Note that $C_2 = a_1^2 + a_2$ is a more sensitive invariant than c_4 in that the residue class of C_2 modulo 3 allows us to distinguish between split and non-split multiplicative reduction, while c_4 does not allow us to separate these two possibilities.

§4. THE CASE $p \geq 5$

Assume $p \geq 5$. Then there exists a minimal Weierstrass equation for E at p of the form

$$(4.1) \quad Y^2 = X^3 + AX + B$$

with $A, B \in \mathbf{Z}$. The coefficient C_{p-1} modulo p is given by Deuring's classical formula [1]

$$(4.2) \quad C_{p-1} \equiv \sum_{2h+3i=P} \frac{P!}{i! h! (P-h-i)!} A^h B^i \pmod{p}$$

where $P = (1/2)(p-1)$.

Let $S = (x, y)$ be the singular point on the reduced curve with $x, y \in \mathbf{Z}/p\mathbf{Z}$. The tangents at S are given by a quadratic polynomial $R(T)$ as follows: Transform the curve by $X \rightarrow (X+x)$, $Y \rightarrow (Y+y)$ so that the singularity is now at $(0, 0)$. The tangents are given by a homogeneous form of degree 2 in X and Y which we can consider as a quadratic polynomial

$R(T)$ with $T = Y/X$. Let D be the discriminant of $R(T)$, and let $\left(\frac{-}{p}\right)$

denote the Legendre symbol with respect to p . We have the following results directly from the definitions.

PROPOSITION 4.1. Assume E has bad reduction at p .

(1) E has additive reduction at $p \Leftrightarrow f_p = 0 \Leftrightarrow S$ is a cusp $\Leftrightarrow R(T)$ has two identical roots over $\mathbf{Z}/p\mathbf{Z} \Leftrightarrow D = 0 \Leftrightarrow \left(\frac{D}{p}\right) = 0$.