4. Growth Estimates for Singular Values

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Two-variable kernels behave very much like their one-variable analogues as regards integrated Lipschitz conditions. Indeed, the following can be easily established:

PROPERTY 4. Kernels in Lip (α, p) also belong to Lip (α, q) for all $1 \leqslant q < p$. Kernels in Lip α are automatically in Lip (α, p) for all $p \geqslant 1$.

PROPERTY 5. Kernels which are relatively uniformly of bounded variation belong to Lip (1, 1).

PROPERTY 6. If $K(x, y)$ is absolutely continuous in x, for almost all y, and

$$\int_0^\pi \left[\int_0^\pi | K^{(1)}(x, y) |^p \, dx\right]^{2/p} dy < \infty,$$

$p > 1$, then $K(x, y)$ is in Lip $(1, p)$.

PROPERTY 7. If a kernel belongs both to Lip (α, p) and to Lip (β, q) with $1 \leqslant p < q$, then it belongs to Lip (γ, r) for all $p \leqslant r \leqslant q$, where

$$\gamma = \alpha \frac{p(q-r)}{r(q-p)} + \beta \frac{q(r-p)}{r(q-p)} .$$

A somewhat deeper result is

PROPERTY 8. Whenever $1 \leqslant p \leqslant q$, $pq(\alpha - \beta) \geqslant q - p$, kernels in Lip (α, p) are automatically also in Lip (β, q).

4. Growth Estimates for Singular Values

We come now to the main thrust of our narrative. The characteristic values associated with a given L^2 kernel $K(x, y)$, $0 \leqslant x, y \leqslant \pi$, are those special values of λ for which there exist nontrivial solutions of the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_0^\pi K(x, y) \phi(y) \, dy .$$

The singular values are those positive values μ for which there exist nontrivial $\phi(x)$, $\Psi(x)$ satisfying the coupled equations

$$\phi(x) = \mu \int_0^\pi K(x, y)\, \Psi(y)\, dy, \tag{4.1}$$
$$\Psi(x) = \mu \int_0^\pi \overline{K(y, x)}\, \phi(y)\, dy.$$

There are at most countably many of each, and they are customarily ordered (indexed) according to increasing modulus, namely

$$0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \dots$$
$$0 < \mu_1 \leqslant \mu_2 \leqslant \dots .$$

The important inequalities (Weyl [32], Chang [8]; see also Gohberg and Krein [16], p. 41, Cochran [11], pp. 243-245)

$$\sum_{n=1}^{N} \left|\frac{1}{\lambda_n}\right|^p \leqslant \sum_{n=1}^{N} \left(\frac{1}{\mu_n}\right)^p \quad p > 0, N \text{ arbitrary,} \tag{4.2}$$

relate their growth behavior.

The earliest known growth estimates concern characteristic values. In 1909, Schur [23] established for continuous kernels that

$$\sum_n \left|\frac{1}{\lambda_n}\right|^2 \leqslant \| K \|^2$$

(This was subsequently extended to L^2 kernels by Carleman [6]). Even earlier, however, Fredholm himself [15] (see also Cochran [10], [11], pp. 251ff.) had essentially shown that

THEOREM 4.1. *If* $K(x, y)$ *is in* Lip α *with* $\alpha > 1/2$, *then*

$$\sum_n \left|\frac{1}{\lambda_n}\right| < \infty .$$

It is interesting to note that this result which, for characteristic values, mirrors Theorem 2.1, actually predated the work of Bernstein.

Numerous other growth estimates, many of them analogous to our earlier Fourier series results of Section 2, have been established by various investigators. Notable among these are the substantial contributions of Hille and Tamarkin [22] (see also Cochran [11], pp. 251-265). For the most part, though, these pertain to characteristic values, and, in view of the

Weyl-Chang inequalities (4.2), the growth behavior of the singular values is of greater intrinsic interest.

With regards to these singular values, we do know that the associated singular functions given by (4.1) can be chosen to be orthonormal amongst themselves as well as biorthogonal with respect to the kernel K. It then follows that

$$K(x, y) = \sum_n \frac{\phi_n(x)\, \overline{\Psi}_n(y)}{\mu_n} \tag{4.3}$$

where the right-hand side converges in the mean. Moreover,

$$\sum_n \left(\frac{1}{\mu_n}\right)^2 = \| K \|^2, \tag{4.4}$$

so that for L^2 kernels we readily conclude that the series of reciprocal singular values is γ-summable at least for all $\gamma \geqslant 2$. The convergence of $\sum (1/\mu_n)^\gamma$ for exponents γ smaller than 2, however, cannot be established without additional restrictions on the kernel K. [1])

The additional restrictions in which we are interested are of the "smoothness" variety. Let us assume that the square-integrable kernel $K(x, y)$ is also such that the $K^{(r)}(x, y)$ $0 \leqslant r \leqslant s - 2$, (defined by (3.1)), are continuous in x, a.e. in y, for some positive (nonnegative) integer s, $K^{(s-1)}(x, y)$ is absolutely continuous in x, a.e. in y, and $K^{(s)}(x, y)$ is in $L^p(x)$, a.e. in y, for some $p > 1$. Under these conditions Smithies [24] essentially showed that

THEOREM 4.2. *If* $K^{(s)}(x, y)$ *belongs to* $\mathrm{Lip}(\alpha, p)$, *then* $\sum (1/\mu_n)^\gamma$ *converges for all* $\gamma > \rho$ *where*

$$\rho = \begin{cases} \dfrac{1}{\alpha + s + 1 - 1/p} & 1 < p \leqslant 2 \\[2ex] \dfrac{1}{\alpha + s + 1/2} & p > 2. \end{cases}$$

When $s = 0$, the additional proviso $\alpha + 1/2 > 1/p$ may be needed since $K \in L^2$.

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1) Although now-a-days it is rather routine to convince yourself of this fact (recall our earlier discussion on difference kernels) Carleman [5] was probably the first to carefully establish that even continuity of the kernel was not generally sufficient to ensure convergence for any $\gamma < 2$.

The Smithies proof is very instructive. As a key ingredient it makes use of the fact that the best mean square approximation to a given square-integrable kernel K by degenerate kernels of the form

$$K_N(x, y) = \sum_{n=1}^{N} a_n(x)\,\bar{b}_n(y) \quad a_n, b_n \in L^2 \tag{4.5}$$

occurs, for fixed N, when the a_n, b_n are proportional to the singular functions ϕ_n, Ψ_n of K [25]. Indeed, if we carry out the details we find

$$\begin{aligned} \left\| K(x, y) - \sum_{n=1}^{N} a_n(x)\,\bar{b}_n(y) \right\|^2 &\leqslant \left\| K(x, y) - \sum_{n=1}^{N} \frac{\phi_n(x)\,\overline{\Psi}_n(y)}{\mu_n} \right\|^2 \\ &= \| K \|^2 - \sum_{n=1}^{N} \left(\frac{1}{\mu_n}\right)^2 \\ &= \sum_{n=N+1}^{\infty} \left(\frac{1}{\mu_n}\right)^2, \end{aligned} \tag{4.6}$$

where we have assumed that the singular functions are orthonormalized and then employed (4.4). In the special case, moreover, when the a_n are the appropriate normalized trigonometric functions, namely $\{\sqrt{2/\pi}\cos nx\}$ if s is even and $\{\sqrt{2/\pi}\sin nx\}$ if s is odd (recall the earlier discussion of Section 3 where we imbued K with certain periodicity properties), and the $\bar{b}_n$ are the resulting Fourier coefficients of $K(x, y)$ viewed as a function of x alone, (4.6) takes the form

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{\mu_n}\right)^2 \leqslant \| K \|^2 - \sum_{n=1}^{N} \int_0^{\pi} | b_n(y) |^2 \, dy.$$

In fact, using essentially Parseval's relation, the right-hand side of this inequality can be further rewritten as

$$\sum_{n=N+1}^{\infty} \left(\frac{1}{\mu_n}\right)^2 \leqslant \sum_{n=N+1}^{\infty} \int_0^{\pi} | b_n(y) |^2 \, dy. \tag{4.7}$$

The intimate relationship that exists between the growth behavior of the singular values associated with two-variable kernels and the asymptotic character of allied classical one-variable Fourier coefficients is rather clearly exhibited by the expression (4.7). This, then, is the essential relationship which engenders the desired analogies. Care must be taken in carrying out the details, however, to ensure that each of the $K^{(r)}$, $0 \leqslant r \leqslant s - 1$, is continuous in the wide-sense, and thus some modification of the behavior of the $K^{(r)}(x, y)$ for $x = 0, \pi$ may be necessary. Fortunately, this can be accomplished with a degenerate perturbation which, as the following lemma shows, leaves unchanged the fundamental asymptotics in question.

LEMMA.[1]) Let $K(x, y), L(x, y), a \leqslant x, y \leqslant b$, be two L^2 kernels which differ by a degenerate kernel, i.e. $K = L + K_N$ where K_N has the form (4.5) for some fixed positive integer N. Then their respective singular values $\mu_n(K)$, $\mu_n(L)$ satisfy

$$\mu_{n-N}(L) \leqslant \mu_n(K) \leqslant \mu_{n+N}(L)$$

for all $n > N$, and hence

$$\mu_n(K) = O(n^\gamma) \text{ iff } \mu_n(L) = O(n^\gamma).$$

Returning to Theorem 4.2, although Smithies didn't use the fact, we note that the special case $s = 0$ is the precise analogue of the Fourier series result Theorem 2.3. In view of Property 4, this case also contains the analogues of the earlier Theorems 2.1, 2.2. Recalling Property 6, moreover, the general case of Theorem 4.2 clearly is analogous to Theorem 2.8 and, as such, actually generalizes to arbitrary $p > 1$ a result alternatively established for $p = 2$ by Smithies' student Chang [9] (see also Gohberg and Krein [16], pp. 119-123).

As in the Fourier series situation, the convexity of the class Lip (α, p) plays an important and extremely useful role. Blending Property 7 with Theorem 4.2, for example, we obtain the following extended analogy to Theorem 2.6:

THEOREM 4.3. *If* $K^{(s)}(x, y)$ *belongs both to* Lip (α, p) *and to* Lip (β, q), *with* $p < q$, *then* $\sum (1/\mu_n)^\gamma$ *converges for all* $\gamma > \rho$ *where* ρ *is as given in Theorem 2.6 but with* α, β *replaced by* $\alpha + s, \beta + s$ *respectively.*

In fashion similar to before, Properties 4, 5 then lead to the special cases

THEOREM 4.4. *If* $K^{(s)}(x, y)$ *is relatively uniformly of bounded variation and also in* Lip (β, q) *for some* $\beta > 0, q \geqslant 1$, *then* $\sum (1/\mu_n)^\gamma$ *converges for all* $\gamma > \rho$ *where* ρ *is as given in Theorem 2.9;*

1) This particular Lemma is a special case of results of Fan [14]. Already in [24], however, Smithies essentially had established the asymptotic invariance property of the singular values under such perturbations.

THEOREM 4.5. *If* $K^{(s)}(x, y)$ *belongs both to* $\mathrm{Lip}(\alpha, p)$ *and to* $\mathrm{Lip}\,\beta$, *then* $\sum (1/\mu_n)^\gamma$ *converges for all* $\gamma > \rho$ *where* ρ *is as given in Theorem 2.10.*

Naturally, these theorems also contain the analogues of the Zygmund and Waraszkiewicz results, Theorems 2.4, 2.5.

In closing it is worth remarking that all of the above kernel function results are equally as sharp as the corresponding Fourier series results since, as we have seen earlier, for periodic difference kernels the singular values and the related Fourier coefficients are essentially reciprocals. In view of the Weyl-Chang inequalities (4.2), moreover, these theorems amplify and extend our knowledge concerning the growth behavior of the *characteristic* values of "smooth" kernels (see [22], [11], for example).

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