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3. PRELIMINARIES

Since in this paper we are concerned for the most part with square-integrable kernels $K(x, y)$, and the Lebesgue integral is employed throughout, equalities and inequalities between functions, therefore, are generally to be understood as holding “almost everywhere” (a.e.).

For convenience we take $a = 0$, $b = \pi$, which we may do without any loss of generality, and consider the class of L^2 kernels $K(x, y)$ with $0 \leq x, y \leq \pi$. In our later work we will need to direct our attention primarily to one of the two variables x, y . Let us choose this to be the first and extend K to be periodic in this variable. There are many ways, of course, to accomplish this task, but a not unreasonable procedure is to first define

$$(3.1) \quad K^{(r)}(x, y) \equiv \frac{\partial^r K(x, y)}{\partial x^r} \quad (r = 0, 1, \dots, s)$$

for some preselected nonnegative integer s , and then assume that $K(x, y)$ is extended, as an even function of x if s is even, and as an odd function of x if s is odd, into the domain $-\pi \leq x \leq 0$, and thence as a periodic function of x with period 2π . This approach ensures that, under suitable restrictions, the classical Fourier series for $K^{(s)}(x, y)$, viewed as a function of its first variable, consists solely of sine terms.

In order to enhance the character of the analogies in which we are interested, we shall say that $K^{(s)}(x, y)$ is in $\text{Lip } \alpha$ if

$$|K^{(s)}(x+h, y) - K^{(s)}(x-h, y)| < |h|^\alpha A(y) \quad (0 < \alpha \leq 1)$$

where $A(y)$ is nonnegative and square-integrable. More generally, for $p \geq 1$, $K^{(s)}(x, y)$ will be said to be in $\text{Lip}(\alpha, p)$ if

$$\int_0^\pi |K^{(s)}(x+h, y) - K^{(s)}(x-h, y)|^p dx < |h|^{\alpha p} A^p(y) \quad (0 < \alpha \leq 1)$$

with $L^2 A \geq 0$. In similar fashion, $K^{(s)}(x, y)$ will be said to be relatively uniformly of bounded variation if for all $N \geq 1$ and arbitrary choice of partition $0 \leq x_0 \leq x_1 \leq \dots \leq x_N \leq \pi$,

$$\sum_{n=1}^N |K^{(s)}(x_n, y) - K^{(s)}(x_{n-1}, y)| < B(y)$$

where $L^2 B \geq 0$. The comparable definitions appropriate whenever the roles of x and y are reversed should be obvious.

Two-variable kernels behave very much like their one-variable analogues as regards integrated Lipschitz conditions. Indeed, the following can be easily established:

PROPERTY 4. Kernels in $\text{Lip}(\alpha, p)$ also belong to $\text{Lip}(\alpha, q)$ for all $1 \leq q < p$. Kernels in $\text{Lip} \alpha$ are automatically in $\text{Lip}(\alpha, p)$ for all $p \geq 1$.

PROPERTY 5. Kernels which are relatively uniformly of bounded variation belong to $\text{Lip}(1, 1)$.

PROPERTY 6. If $K(x, y)$ is absolutely continuous in x , for almost all y , and

$$\int_0^\pi \left[\int_0^\pi |K^{(1)}(x, y)|^p dx \right]^{2/p} dy < \infty,$$

$p > 1$, then $K(x, y)$ is in $\text{Lip}(1, p)$.

PROPERTY 7. If a kernel belongs both to $\text{Lip}(\alpha, p)$ and to $\text{Lip}(\beta, q)$ with $1 \leq p < q$, then it belongs to $\text{Lip}(\gamma, r)$ for all $p \leq r \leq q$, where

$$\gamma = \alpha \frac{p(q-r)}{r(q-p)} + \beta \frac{q(r-p)}{r(q-p)}.$$

A somewhat deeper result is

PROPERTY 8. Whenever $1 \leq p \leq q$, $pq(\alpha - \beta) \geq q - p$, kernels in $\text{Lip}(\alpha, p)$ are automatically also in $\text{Lip}(\beta, q)$.

4. GROWTH ESTIMATES FOR SINGULAR VALUES

We come now to the main thrust of our narrative. The characteristic values associated with a given L^2 kernel $K(x, y)$, $0 \leq x, y \leq \pi$, are those special values of λ for which there exist nontrivial solutions of the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_0^\pi K(x, y) \phi(y) dy.$$

The singular values are those positive values μ for which there exist nontrivial $\phi(x)$, $\Psi(x)$ satisfying the coupled equations