

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	22 (1976)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
Artikel:	SUMMABILITY OF SINGULAR VALUES OF L^2 KERNELS. ANALOGIES WITH FOURIER SERIES
Autor:	Cochran, James Alan
Kapitel:	2. Fourier Séries Results
DOI:	https://doi.org/10.5169/seals-48180

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 22.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

2. FOURIER SERIES RESULTS

Let the integrable function $f(x)$, $-\pi \leq x \leq \pi$, have period 2π , so that $f(x+2\pi) = f(x)$, and in particular $f(\pi) = f(-\pi)$, and assume that $0 < \alpha \leq 1$ and $p \geq 1$. Denote by Δf one of the three differences (it matters not which for our purposes)

$$f(x) - f(x-h), \quad f(x+h) - f(x), \quad f(x+h) - f(x-h).$$

If $\Delta f = O(|h|^\alpha)$ we say either that $f(x)$ *belongs to* $\text{Lip } \alpha$ or that $f(x)$ *satisfies a Lipschitz condition with exponent* α . More generally, $f(x)$ is said to *belong to the Lipschitz class* $\text{Lip } (\alpha, p)$ if

$$\int_{-\pi}^{\pi} |\Delta f|^p dx = O(|h|^{\alpha p}).$$

In view of Hölder's inequality, a function of $\text{Lip } (\alpha, p)$ belongs to $\text{Lip } (\alpha, q)$ for all $1 \leq q < p$. Moreover, a function of $\text{Lip } \alpha$ clearly belongs to $\text{Lip } (\alpha, p)$ for all $p \geq 1$. In fact, the class $\text{Lip } \alpha$ may be viewed roughly as the limit of $\text{Lip } (\alpha, p)$ for $p = \infty$.

The classical complex Fourier series of $f(x)$ is defined by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \text{ where } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Equivalently, if $c_n \equiv \frac{1}{2} (a_n - ib_n)$ for all n , then

$$f(x) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

with

$$\frac{a_n}{b_n} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{\cos}{\sin} nx dx.$$

For given integrable f , the series

$$\sum_{n=-\infty}^{\infty} |c_n|^\gamma$$

of moduli of the coefficients of these Fourier series may not converge for any finite $\gamma > 0$. If it does for certain γ , however, the convergence exponent

ρ of the Fourier coefficients is the infimum of these γ . For square-integrable f , we know that $\rho \leq 2$. (The above series, of course, need not be convergent for $\gamma = \rho$.)

The earliest result of interest to us here is the well-known theorem of Bernstein [2], [3], [4] (see also Bary [1], pp. 153-171, or Zygmund [37], pp. 240-243, for example) which we state as follows:

THEOREM 2.1. *If $f(x)$ is in $\text{Lip } \alpha$ with $\alpha > \frac{1}{2}$, then $\rho < 1$.*

This result has a sharpened form due to Szász [26], namely:

THEOREM 2.2. *If $f(x)$ is in $\text{Lip } \alpha$, then $\rho = 1/(\alpha + 1/2)$,*

and an even more general rendition due essentially to Szász [26] (the case $p = 2$), [27], Titchmarsh [28] (the corresponding theorem for transforms; see also [29]), and Hardy and Littlewood [19] (under the assumption $\alpha p > 1$):

THEOREM 2.3. *If $f(x)$ belongs to $\text{Lip } (\alpha, p)$, then*

$$\rho = \begin{cases} \frac{1}{\alpha + 1 - 1/p} & 1 \leq p \leq 2 \\ \frac{1}{\alpha + 1/2} & p > 2. \end{cases}$$

For square-integrable f , this result only has relevance, of course, when $2\alpha p > 2 - p$.

We note in passing that since the class $\text{Lip } (1, p)$, where $p > 1$, is equivalent to the collection of integrals of functions of the Lebesgue class L^p (Hardy and Littlewood [18], p. 599), Theorem 2.3 has as a special case the well-known result originally established by Tonelli [30]:

COROLLARY. *If $f(x)$ is absolutely continuous and its derivative $f'(x)$ belongs to L^p , $p > 1$, then $\rho < 1$.*

Other restrictions on $f(x)$, $-\pi \leq x \leq \pi$, are also of interest to us. Finite-valued functions are said to be of *bounded variation* if for all $N \geq 1$ and arbitrary choice of partition $-\pi \leq x_0 \leq x_1 \leq \dots \leq x_N \leq \pi$,

$$\sum_{n=1}^N |f(x_n) - f(x_{n-1})| \leq B \text{ (const.)} < \infty.$$

Since $f(x)$ is in $\text{Lip}(1, 1)$ if and only if (iff) it is of bounded variation, no new results arise without at least some modest additional assumptions beyond mere bounded variation. One such set of combined restrictions leads to the following classical result first established by Zygmund [35] (see also Bary [1], Zygmund [37]):

THEOREM 2.4. *If $f(x)$ is of bounded variation and also in $\text{Lip } \beta$ for some $\beta > 0$, then $\rho < 1$.*

Here also there is a sharpened form, this time due to Waraszkiewicz [31] (see also Zygmund [36]):

THEOREM 2.5. *If $f(x)$ is of bounded variation and also in $\text{Lip } \beta$ for some $\beta > 0$, then $\rho = 1/(1 + \beta/2)$.*

Other results, employing different sets of combined assumptions, can be established using the convexity property of the class $\text{Lip}(\alpha, p)$ (Hardy and Littlewood [20]), namely:

PROPERTY 1. If $f(x)$ belongs both to $\text{Lip}(\alpha, p)$ and to $\text{Lip}(\beta, q)$, where $p < q$, then it belongs to $\text{Lip}(\gamma, r)$ for all $p \leq r \leq q$, where

$$\gamma = \alpha \frac{p(q-r)}{r(q-p)} + \beta \frac{q(r-p)}{r(q-p)}.$$

In the limiting case $q = \infty$, where $f(x)$ is in $\text{Lip } \beta$, then

$$\gamma = \beta + (\alpha - \beta) \frac{p}{r}.$$

Interplaying this property with the earlier Theorem 2.3, we obtain the general

THEOREM 2.6. *If $f(x)$ belongs both to $\text{Lip}(\alpha, p)$ and to $\text{Lip}(\beta, q)$, where $p < q$, then*

i) *for $q \leq 2$,*

$$\rho = \begin{cases} \frac{1}{\alpha + 1 - 1/p} & pq(\alpha - \beta) > q - p \\ \frac{1}{\beta + 1 - 1/q} & pq(\alpha - \beta) \leq q - p, \end{cases}$$

ii) for $p \leq 2 < q$,

$$\rho = \begin{cases} \frac{1}{\alpha + 1 - 1/p} & pq(\alpha - \beta) > q - p \\ \frac{2(q-p)}{q(2\beta + \alpha p + 1) - p(2\alpha + \beta q + 1)} & 0 < pq(\alpha - \beta) \leq q - p \\ \frac{1}{\beta + 1/2} & \alpha \leq \beta, \end{cases}$$

iii) and for $p > 2$,

$$\rho = \begin{cases} \frac{1}{\alpha + 1/2} & \alpha > \beta \\ \frac{1}{\beta + 1/2} & \alpha \leq \beta. \end{cases}$$

Theorem 2.5 is the special case of this result when $\alpha = p = 1, q = \infty$. Other special cases are:

COROLLARY 1. If $f(x)$ is of bounded variation and also in $\text{Lip}(\beta, q)$ for some $\beta > 0, q \geq 1$, then

$$\rho = \begin{cases} 1 & \beta q < 1 \\ \frac{q}{\beta q + q - 1} & \beta q \geq 1, q \leq 2 \\ \frac{2(q-1)}{\beta q + 2q - 3} & \beta q \geq 1, q > 2; \end{cases}$$

COROLLARY 2. If $f(x)$ belongs to $\text{Lip}(\alpha, p)$ and also satisfies an ordinary Lipschitz condition with exponent $\beta > 0$, then

$$\rho = \begin{cases} \frac{p}{\alpha p + p - 1} & p(\alpha - \beta) > 1, \quad p \leq 2 \\ \frac{2}{\beta(2-p) + \alpha p + 1} & 0 < p(\alpha - \beta) \leq 1, \quad p \leq 2 \\ \frac{1}{\alpha + 1/2} & \alpha > \beta, \quad p > 2 \\ \frac{1}{\beta + 1/2} & \alpha \leq \beta. \end{cases}$$

We note that $\rho < 1$ for $\beta q > 1$ in the first case, while for $p \leq 2$, $\alpha > \beta > (1 - \alpha p)/(2 - p)$ gives the same conclusion in the latter situation. Comparable results were observed by Hardy and Littlewood [20] and Waraszkiewicz [31].

Perhaps not surprisingly, the Corollary to Theorem 2.3 may be viewed as a special case of Corollary 2 above since when $\alpha p > 1$, functions in $\text{Lip}(\alpha, p)$ likewise belong to $\text{Lip}(\alpha - 1/p + 1/q, q)$ for all $q > p$ and hence are equivalent to functions in $\text{Lip}(\alpha - 1/p)$ (Hardy and Littlewood [19]). Alternatively, the earlier result can also be established using the following variant of one-half of the Hausdorff-Young Theorem (Hausdorff [21], Young [33], [34]; see also Hardy and Littlewood [17], Bary [1], Zygmund [37]) and the familiar relation between the Fourier coefficients of $f(x)$ and its derivatives $f^{(s)}(x)$, $s = 1, 2, \dots$:

THEOREM 2.7. *If $f(x)$ is in L^p , $p > 1$, then*

$$\rho = \begin{cases} \frac{p}{p-1} & p \leq 2 \\ 2 & p > 2. \end{cases}$$

PROPERTY 2. If $f^{(s-1)}(x)$ is absolutely continuous for some positive integer s , then the Fourier coefficients c_{ns} of $f^{(s)}(x)$ are given by

$$c_{ns} = (in)^s c_n.$$

(Here, of course, we have made the tacit assumption that the periodic $f^{(r)}(x)$, $0 \leq r \leq s-1$, are all continuous in the wide-sense, i.e. for all x , so that in particular $f^{(r)}(\pi) = f^{(r)}(-\pi)$, $0 \leq r \leq s-1$.) Property 2 easily leads to

PROPERTY 3. If $f^{(s-1)}(x)$ is absolutely continuous for some positive integer s , and the convergence exponent of the Fourier coefficients of $f^{(s)}(x)$ is ρ_s , then

$$\rho = \frac{\rho_s}{1 + s\rho_s}.$$

Taken together, the above results finally yield the general

THEOREM 2.8. *If $f^{(s-1)}(x)$ is absolutely continuous for some positive integer s , and $f^{(s)}(x)$ belongs to L^p , $p > 1$, then*

$$\rho = \begin{cases} \frac{p}{p(s+1)-1} & p \leq 2 \\ \frac{2}{1+2s} & p > 2. \end{cases}$$

In particular, for $s = 1$

$$\rho = \begin{cases} \frac{p}{2p-1} & p \leq 2 \\ \frac{2}{3} & p > 2. \end{cases}$$

Any number of other deductions can be obtained by combining Theorem 2.8 with earlier results. We content ourselves with

THEOREM 2.9. *If $f^{(s-1)}(x)$ is absolutely continuous for some positive s , and if $f^{(s)}(x)$ is of bounded variation and also in $\text{Lip}(\beta, q)$ for some $\beta > 0$, $q \geq 1$, then*

$$\rho = \begin{cases} \frac{1}{1+s} & \beta q < 1 \\ \frac{q}{q(\beta+1+s)-1} & \beta q \geq 1, q \leq 2 \\ \frac{2(q-1)}{q(\beta+2+2s)-3-2s} & \beta q \geq 1, q > 2; \end{cases}$$

THEOREM 2.10. *If $f^{(s-1)}(x)$ is absolutely continuous for some positive s , and if $f^{(s)}(x)$ belongs to $\text{Lip}(\alpha, p)$ and also satisfies an ordinary Lipschitz condition with exponent $\beta > 0$, then*

$$\rho = \begin{cases} \frac{p}{p(\alpha+1+s)-1} & p(\alpha-\beta) > 1, p \leq 2 \\ \frac{2}{\beta(2-p)+\alpha p+1+2s} & 0 < p(\alpha-\beta) \leq 1, p \leq 2 \\ \frac{1}{\alpha+s+1/2} & \alpha > \beta, p > 2 \\ \frac{1}{\beta+s+1/2} & \alpha \leq \beta. \end{cases}$$