Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 22 (1976)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SUMMABILITY OF SINGULAR VALUES OF L^2 KERNELS. ANALOGIES WITH FOURIER SERIES

Autor: Cochran, James Alan

Kapitel: 1. Introduction

DOI: https://doi.org/10.5169/seals-48180

SUMMABILITY OF SINGULAR VALUES OF L^2 KERNELS. ANALOGIES WITH FOURIER SERIES

by James Alan COCHRAN

1. INTRODUCTION

The exploitation of analogies between related mathematical contructs is an often-fruitful endeavor. A thorough grounding in finite-dimensional vector spaces enhances the mastery of Hilbert space concepts; knowledge of the characteristic behavior of harmonic functions suggests properties which can be shown to be shared by solutions of far more general elliptic partial differential equations; the convergence question for a given infinite series is made clear through investigation of a related improper integral. Other examples abound, including the reader's own personal favorite.

In this paper we shall be concerned with L^2 kernels, i.e. two-variable functions $K(x, y)$ defined for $a \leqslant x, y \leqslant b$ and satisfying

$$\| K \| \equiv [\int_a^b \int_a^b | K(x, y) |^2 \, dx \, dy]^{\frac{1}{2}} < \infty,$$

which are envisioned as the kernels of linear Fredholm integral equations. As is customary, we term the nonnegative square roots of the characteristic values of the related kernel $KK^* (x, y)$, the singular values μ_n of the original kernel. Our specific interest is in the connection between the *smoothness* of the given kernel K and the *growth behavior* of these singular values. More particularly, we explore, illuminate, and in general "exploit" the remarkable analogies that prevail between this growth behavior of singular values associated with square-integrable kernels satisfying various smoothness criteria and the values of convergence exponents for classical Fourier series under comparable conditions.

The existence of at least some sort of relationship which permits these analogies is certainly to be suspected in view of the parallelism of the well-known Fourier series result of M. Riesz (see Hardy and Littlewood [20], Bary [1], pp. 184ff, or Zygmund [37], p. 251) that "the Fourier series of an L^2 function f converges absolutely if and only if f can be represented as the convolution of two other L^2 functions," on the one hand, and the nuclear kernel result (Chang [7], [8]; see also Cochran [11], pp. 236-237) that "the series of reciprocal singular values of an L^2 kernel K converges (absolutely, of course) if and only if K can be represented as the composition of two other L^2 kernels," on the other. Especially venturesome readers might even be willing to conjecture such a relationship merely on the basis of the considerable use, over the years, of periodic functions of one variable to generate difference kernels of two variables having specified properties. The carry over of growth/smoothness connections, of course, is immediate in these special cases. Indeed, we need only recall that if $f(x)$, $-\pi \leqslant x \leqslant \pi$, is square-integrable, periodic with period 2π, and has the classical Fourier series coefficients c_n, then the correspondence

$$K(x, y) \equiv f(x-y) \quad -\pi \leqslant x, y \leqslant \pi$$

leads to a (normal) kernel with singular values

$$\mu_n = 1/2\pi \mid c_n \mid .$$

We should expect the analysis of the general situation to be considerably more complicated, however.

Perhaps somewhat surprisingly then it actually turns out that the specific relationship which makes possible the general analogies which are the subject of this paper is not an exceedingly deep result, when viewed in the appropriate context, and we shall consider it carefully in a later section. For the present we merely note that the relationship was essential for an investigation carried out by Smithies and reported on already in 1937 [24]. Since the harvest is so rich, we can only conjecture why the relationship lay fallow for so many years and only recently was "rediscovered" and put to full use [13].

In the next section of this paper we list the various classical Fourier series results with which we shall be concerned. These include the several sufficiency conditions for absolute convergence of Fourier series due to Bernstein and Zygmund, for example, as well as numerous more precise results of Hardy and Littlewood, Szász, and others. Subsequently, in Section 3, we gather together the mathematical machinery needed for the investigation of the analogous spectral-theoretic results. A full discussion of the growth behavior of the singular values for the various kernel smoothness conditions of interest is then given in Section 4, along with some additional historical perspective.