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This follows from Theorem 2: the right-hand side is at most a multiple of the left since  $F(x, t) \rightarrow F(x, 0)$  in  $L_w^p$ ; the converse inequality is just (3) with  $w_2$  and  $r$  chosen to be  $w$  and  $p$ , resp.

### §3. PROOF OF THEOREMS 1 AND 3

We will prove Theorem 1 first, beginning with part (i). Let  $F \in H_w^1$ ,  $F = (u, v_1, \dots, v_n)$ ,  $w \in A_1$ . By Theorem 2,  $F$  has boundary values  $F(x, 0) = (f(x), g_1(x), \dots, g_n(x))$  pointwise a.e. and in  $L_w^1$ . In particular,  $f, g_1, \dots, g_n \in L_w^1$ . We will show that  $u = P(f)$  and  $v_j = P(g_j)$ . Since  $u(x, s)$  converges to  $f(x)$  in  $L_w^1$ ,  $P(u(\cdot, s))(x, t) \rightarrow (Pf)(x, t)$  as  $s \rightarrow 0$ :

$$\begin{aligned} |P(u(\cdot, s))(x, t) - (Pf)(x, t)| &= \left| \int_{R^n} [u(y, s) - f(y)] P(x - y, t) dy \right| \\ &\leq \|u(\cdot, s) - f\|_{1, w} \left\{ \sup_y w(y)^{-1} P(x - y, t) \right\}, \end{aligned}$$

where the expression in curly brackets is finite for each  $(x, t)$  (see (6)). By Lemma 1,  $u(x, s+t) = P(u(\cdot, s))(x, t)$  since  $u$  is harmonic. Hence, letting  $s \rightarrow 0$ , we obtain  $u(x, t) = (Pf)(x, t)$ , as desired. The argument proving that  $v_j = P(g_j)$  is similar.

Now let  $G = (Pf, Q_1 f, \dots, Q_n f)$ . Then  $G$  is a Cauchy-Riemann system with the same first component as  $F$ . This implies that the first component of  $F - G$  is zero, and so that the others are independent of  $t$ ; that is,  $v_j - Q_j f$  is independent of  $t$ . Thus,  $v_j = Q_j f$  if both  $v_j(x, t)$  and  $(Q_j f)(x, t)$  tend to zero as  $t \rightarrow +\infty$  ( $x$  fixed). We have already observed this for  $Q_j f$ . For  $v_j$ , the mean-value property of harmonic functions gives

$$\begin{aligned} |v_j(x, t)| &\leq ct^{-n-1} \iint_{|\xi-x|^2 + |t-\eta|^2 < t^2} |v_j(\xi, \eta)| d\xi d\eta \\ &\leq ct^{-n} \sup_{\eta > 0} \int_{|\xi-x| < t} |v_j(\xi, \eta)| d\xi \\ &\leq ct^{-n} \left( \sup_{\eta > 0} \int_{R^n} |v_j(\xi, \eta)| w(\xi) d\xi \right) \left( \sup_{\xi: |\xi-x| < t} w(\xi)^{-1} \right) \\ &\leq ct^{-n} \sup_{\xi: |\xi-x| < t} w(\xi)^{-1}. \end{aligned}$$

Since  $w(\xi)^{-1} \leq c(1+|\xi|)^{n\delta}$  for some  $\delta$ ,  $0 < \delta < 1$ , we have

$$|v_j(s, t)| \leq ct^{-n}(1+|x|+t)^{n\delta}.$$

Hence,  $v_j(x, t) \rightarrow 0$  as  $t \rightarrow \infty$  for each  $x$ .

We now know  $u = Pf$ ,  $v_j = P(g_j) = Q_jf$ . Letting  $t \rightarrow 0$  in the equation  $P(g_j)(x, t) = (Q_jf)(x, t)$  gives  $g_j(x) = (R_jf)(x)$  a.e. Thus,  $R_jf \in L_w^1$  and  $v_j = P(R_jf) = Q_jf$ , as desired. All that remains to prove in (i) is that  $\|F\|$  and  $\|f\|_{1,w} + \sum_{j=1}^n \|R_jf\|_{1,w}$  are equivalent. This, however, follows immediately from (10) with  $p = 1$ , since

$$F(x, 0) = (f(x), R_1f(x), \dots, R_nf(x)).$$

To prove (ii), let  $f$  be a function in  $L_w^1$  for which each  $R_jf \in L_w^1$ . (The existence of  $R_jf$  as a pointwise limit is guaranteed by the hypothesis  $w \in A_1$ .) We will show that the vector defined by

$$F = (Pf, Q_1f, \dots, Q_nf)$$

is in  $H_w^1$ . Once this is done, the rest of (ii) clearly follows from (i). We know  $F$  is a Cauchy-Riemann system, and only need to show  $\|F\| < +\infty$ . As  $t \rightarrow 0$ ,  $F(x, t)$  converges a.e. to  $(f, R_1f, \dots, R_nf) = F(x, 0)$ , say, so that  $|F(x, 0)| \in L_w^1$ . Hence,  $\|F\| < +\infty$  by Theorem 2 if there exist  $p$  and  $w_1$ ,  $\frac{n-1}{n} < p < \infty$ ,  $w_1 \in A_{pn/(n-1)}$ , such that

$$(11) \quad \sup_{t>0} \int_{R^n} |F(x, t)|^p w_1(x) dx < +\infty.$$

We first claim that if  $w \in A_1$ , there exists  $\alpha > 0$  such that the function

$$w_1(x) = \frac{w(x)}{(1+|x|)^\alpha}$$

also belongs to  $A_1$ . Note that  $(1+|x|)^{-\beta} \in A_1$  if  $0 \leq \beta < n$ , and that there exists  $s > 1$  such that  $w^s \in A_1$ . Hence, for any cube  $I$ , Hölder's inequality gives

$$\frac{1}{|I|} \int_I w_1(x) dx \leq \left( \frac{1}{|I|} \int_I w(x)^s dx \right)^{1/s} \left( \frac{1}{|I|} \int_I (1+|x|)^{-\alpha s'} dx \right)^{1/s'},$$

$s' = s/(s-1)$ . Choose  $\alpha > 0$  so small that  $\alpha s' < n$ . Then both  $w^s$  and  $(1+|x|)^{-\alpha s'}$  are in  $A_1$ , and

$$\begin{aligned} \frac{1}{|I|} \int_I w_1(x) dx &\leq c (\operatorname{ess\,inf}_I w^s)^{1/s} (\operatorname{ess\,inf}_I (1+|x|)^{-\alpha s'})^{1/s'} \\ &= c (\operatorname{ess\,inf}_I w) (\operatorname{ess\,inf}_I (1+|x|)^{-\alpha}) \\ &\leq c \operatorname{ess\,inf}_I w_1. \end{aligned}$$

This proves the claim.

With this choice of  $w_1$ , we will complete the proof of (ii) by showing that (11) holds for any  $p < 1$  which is sufficiently close to 1. Let

$$(R^*f)(x) = \max_{j=1, \dots, n} (R_j^*f)(x).$$

Then, as is well-known, there is a constant  $c$  depending only on  $n$  such that

$$|F(x, t)| \leq c [f^*(x) + (R^*f)(x)].$$

It follows from the weak-type estimates referred to in §2 that the radial maximal function  $N_0(F)(x) (= \sup_{t>0} |F(x, t)|)$  satisfies

$$m_w \{x: N_0(F)(x) > \lambda\} \leq c \lambda^{-1} \|f\|_{1,w}, \quad \lambda > 0.$$

We will show that any non-negative function  $\phi$  with

$$m_w \{x: \phi(x) > \lambda\} \leq c \lambda^{-1}, \quad \lambda > 0,$$

belongs to  $L_{w_1}^p$ ,  $1 - \frac{\alpha}{n} < p < 1$ . Let  $g_r(\lambda)$ ,  $\lambda > 0$ , denote the non-increasing rearrangement of a function  $g$  with respect to the measure  $w(x) dx$ . Then, by [5], p. 257,

$$\begin{aligned} \int_{R^n} \phi^p w_1 dx &= \int_{R^n} \phi(x)^p (1+|x|)^{-\alpha} w(x) dx \\ &\leq \int_0^\infty \phi_r^p(\lambda) \{(1+|x|)^{-\alpha}\}_r(\lambda) d\lambda. \end{aligned}$$

We have  $\phi_r(\lambda) \leq c \lambda^{-1}$  and must estimate  $\{(1+|x|)^{-\alpha}\}_r$ . However,

$$m_w \{x: (1+|x|)^{-\alpha} > \lambda\} = m_w \{x: 1+|x| < \lambda^{-1/\alpha}\},$$

which for  $\lambda \geq 1$  is zero and for  $0 < \lambda < 1$  is less than

$$\int_{|x| < \lambda^{-1/\alpha}} w dx \leq c \lambda^{-n/\alpha} \int_{|x| < 1} w dx = c \lambda^{-n/\alpha}$$

(see (5)). Therefore,

$$\{(1+|x|)^{-\alpha}\}_r(\lambda) \leq c(1+\lambda)^{-\alpha/n}, \lambda > 0.$$

Combining estimates, we obtain

$$\int_{R^n} \phi^p w_1 dx \leq c \int_0^\infty \lambda^{-p} (1+\lambda)^{-\alpha/n} d\lambda < +\infty$$

if  $1 - \frac{\alpha}{n} < p < 1$ , as desired. This completes the proof of (ii).

To prove Theorem 3, let  $f \in L_w^1$  and  $w \in A_1$ . Then (11) holds for  $F$ ,  $p$  and  $w_1$  as in the proof of Theorem 1 (ii). (The proof of (11) does not require  $R_j f \in L_w^1$ .) Hence, by Lemma 2 (see (8)),

$$N(F)(x) \leq c(|F(x, 0)|^{\frac{n-1}{n}})^{* \frac{n}{n-1}}.$$

Since  $F(x, 0) = (f(x), (R_1 f)(x), \dots, (R_n f)(x))$ , the conclusion of Theorem 3 follows immediately with  $\mu = (n-1)/n$ .

To prove the fact stated at the end of the introduction, let

$$f, R_1 f, \dots, R_n f \in L^1.$$

Clearly,

$$\begin{aligned} P(R_j f)^{\wedge}(x, t) &= \hat{P}(x, t)(R_j f)^{\wedge}(x) = e^{-2\pi t|x|}(R_j f)^{\wedge}(x), \\ (Q_j f)^{\wedge}(x, t) &= \hat{Q}_j(x, t)\hat{f}(x) = i \frac{x_j}{|x|} e^{-2\pi t|x|} \hat{f}(x) \text{ a.e.}, \end{aligned}$$

where the Fourier transform is taken in the  $x$  variable with  $t$  fixed. (Note that for fixed  $t$ ,  $P(x, t)$  belongs to  $L^1$  and  $Q_j(x, t)$  belongs to  $L^2$ .) However, these expressions are all equal everywhere since  $P(R_j f) = Q_j f$  by Theorem 1 and  $P(R_j f) \in L^1$ . Therefore,  $(R_j f)^{\wedge}(x) = ix_j |x|^{-1} \hat{f}(x)$ , as claimed.

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