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A BOUNDARY VALUE CHARACTERIZATION OF WEIGHTED H^1

by Richard L. WHEEDEN [1])

ABSTRACT

We give a proof of the elementary result that for certain weight functions w, the Hardy space H_w^1 can be identified with the class of functions f such that f and all its Riesz transforms $R_j f$ belong to L_w^1. An important ingredient of the proof is that there exist positive constants c and μ, $0 < \mu < 1$, depending only on the dimension n such that if f belongs to L_w^1, then

$$N(f)(x) \leqslant c\left[M_\mu(f)(x) + \sum_{j=1}^{n} M_\mu(R_j f)(x)\right],$$

where $N(f)$ denotes the non-tangential maximal function of the Poisson (or any conjugate Poisson) integral of f, and M_μ denotes the Hardy-Littlewood maximal operator of order μ:

$$M_\mu(g)(x) = \left(\sup_{h>0} h^{-n} \int_{|y|<h} |g(x+y)|^\mu \, dy\right)^{1/\mu}.$$

§1. INTRODUCTION

Let $F(x,t) = (u(x,t), v_1(x,t), \ldots, v_n(x,t))$, $x = (x_1, \ldots, x_n) \in R^n$, $t > 0$, satisfy the Cauchy-Riemann equations in the sense of Stein and Weiss [9]: i.e., $u, v_1, \ldots, v_n$ are harmonic in

$$R_+^{n+1} = \{(x,t) : x \in R^n, \ t > 0\}, \ \frac{\partial u}{\partial t} + \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} = 0$$

and

$$\frac{\partial v_j}{\partial x_i} = \frac{\partial v_i}{\partial x_j}, \ \frac{\partial v_j}{\partial t} = \frac{\partial u}{\partial x_j}$$

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there. F is said to belong to H_w^1, where w is a non-negative measurable function on R^n, if

$$|||F||| = \sup_{t>0} \int_{R^n} |F(x,t)|\, w(x)\, dx < +\infty .$$

Letting

$$L_w^1 = \left\{ f:\ ||f||_{1,w} = \int_{R^n} |f(x)|\, w(x)\, dx < +\infty \right\},$$

it is immediate that a Cauchy-Riemann system belongs to H_w^1 if and only if the L_w^1 norms of its components are uniformly bounded for $t > 0$. See also [4], p. 118.

We consider primarily weight functions w satisfying

$$(A_1) \qquad \frac{1}{|I|} \int_I w(x)\, dx \leqslant c \operatorname*{ess\,inf}_I w ,$$

where I is a "cube" in R^n, and c is a constant independent of I. (See [7], [3].) If $w \in A_1$ and $f \in L_w^1$, the Riesz transforms of f, defined as the *pointwise* limits

$$(1) \qquad \begin{aligned} &(R_j f)(x) = \lim_{\varepsilon \to 0} (R_{j,\varepsilon} f)(x),\ j = 1, \ldots, n, \quad \text{where} \\ &(R_{j,\varepsilon} f)(x) = c_n \int_{|y|>\varepsilon} f(x-y) \frac{y_j}{|y|^{n+1}}\, dy,\ c_n = \Gamma\left(\frac{n+1}{2}\right) / \pi^{\frac{n+1}{2}}, \end{aligned}$$

exist a.e. (See [2].) Moreover, as we shall see, the Poisson and conjugate Poisson integrals of f exist and are finite if $f \in L_w^1$, $w \in A_1$. These will be denoted respectively by

$$(Pf)(x,t) = \int_{R^n} f(x-y)\, P(y,t)\, dy ,$$

$$(Q_j f)(x,t) = \int_{R^n} f(x-y)\, Q_j(y,t)\, dy ,$$

where

$$P(y,t) = c_n t / (t^2 + |y|^2)^{\frac{n+1}{2}}$$

and

$$Q_j(y,t) = c_n y_j / (t^2 + |y|^2)^{\frac{n+1}{2}}$$

are the Poisson and conjugate Poisson kernels. The vector $(Pf, Q_1 f, ..., Q_n f)$ is of course a Cauchy-Riemann system, and the formulas

$$\lim_{t\to 0} (Pf)(x,t) = f(x), \lim_{t \to 0} (Q_j f)(x,t) = (R_j f)(x)$$

hold a.e. if $f \in L^1_w$, $w \in A_1$.

THEOREM 1. *Let* $w \in A_1$.

(i) *If* $F = (u, v_1, ..., v_n)$ *belongs to* H^1_w, *there exists* $f \in L^1_w$ *such that* $R_j f \in L^1_w$, $u = Pf$ *and* $v_j = P(R_j f) = Q_j f$ *for each* j. *Moreover, there are positive constants* c_1 *and* c_2, *independent of* F, *such that*

$$c_1 |||F||| \leqslant ||f||_{1,w} + \sum_1^n ||R_j f||_{1,w} \leqslant c_2 |||F||| . \tag{2}$$

(ii) *Let* $f \in L^1_w$. *If each* $R_j f \in L^1_w$, *then the vector*

$$F = (Pf, Q_1 f, ..., Q_n f)$$

belongs to H^1_w. *Moreover,* $Q_j f = P(R_j f)$ *and* (2) *holds.*

Thus, if $w \in A_1$, H^1_w can be identified with

$$\{f: ||f|| = ||f||_{1,w} + \sum_1^n ||R_j f||_{1,w} < +\infty\},$$

with equivalence of norms. This result, which is very natural, seems to be generally taken for granted, although there appear to be no proofs (at least of (ii)) in the literature, even when $w \equiv 1$. In the one-dimensional periodic case with $w \equiv 1$, two proofs of (ii) are given in [11], vol. 1: see (4.4), p. 263, and the remark at the bottom of p. 285. Our proof is modelled after the second of these. It is largely technical and contains little that is new; a simpler proof would be interesting. The proof of (i) is fairly standard and included only for completeness.

A weight w is said to belong to A_p, $1 < p < \infty$, if there is a constant c such that

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} \leqslant c \tag{A_p}$$

for all cubes I. (See [7]). For $0 < p < \infty$, let

$$L^p_w = \left\{f: ||f||_{p,w} = \left(\int_{R^n} |f|^p w\,dx\right)^{1/p} < +\infty\right\}.$$

In the course of proving (ii), we will derive the following result, analogous to Theorem D of [9], about boundary values of Cauchy-Riemann systems.

THEOREM 2. *Let* F *be a Cauchy-Riemann system for which*

$$\sup_{t>0} \int_{R^n} |F(x,t)|^p w_1(x)\,dx < +\infty ,$$

where $\frac{n-1}{n} < p < \infty$ *and* $w_1 \in A_{pn/(n-1)}$. *Then* $F(x,t)$ *has a limit* $F(x,0)$ *a.e. (and in* $L^p_{w_1}$*) as* $t \to 0$. *If* $|F(x,0)| \in L^r_{w_2}$ *for* $\frac{n-1}{n} < r < \infty$ *and* $w_2 \in A_{rn/(n-1)}$, *there is a constant* c, *depending only on* n *and* w_1, *such that*

$$\sup_{t>0} \int_{R^n} |F(x,t)|^r w_2(x)\,dx \leqslant c\, \| F(x,0) \|^r_{r,w_2} . \tag{3}$$

The case $w_1 = w_2 = 1$ is proved in [9].

It follows (see (10) below) that for $F \in H^1_w$, $|||F|||$ and $\|F(x,0)\|_{1,w}$ are equivalent if $w \in A_{n/(n-1)}$. This is an interesting contrast to Theorem 1, which gives more boundary information, but requires the stronger condition $w \in A_1$.

The method used to prove Theorems 1 and 2 leads to the following result, in which we use the notation

$$N(F)(x) = \sup \{ |F(y,t)| : (y,t) \text{ satisfies } |x-y| < t \},$$

$$(M_\mu f)(x) = \left(\sup_{h>0} h^{-n} \int_{|y|<h} |f(x+y)|^\mu dy \right)^{1/\mu}, \mu > 0 .$$

THEOREM 3. *Let* f *belong to* L^1_w, *and let* w *satisfy* A_1. *Let* $F = (Pf, Q_1 f, ..., Q_n f)$. *There exist positive constants* c *and* μ *depending only on* n *such that* $0 < \mu < 1$ *and*

$$N(F)(x) \leqslant c\,[M_\mu(f)(x) + \sum_{j=1}^n M_\mu(R_j f)(x)] .$$

The constant μ above can be taken to be $(n-1)/n$. It follows easily from this and the results of [7] that if $f \in L^1_{w_1}$ for *any* w_1 satisfying A_1

and if $f, R_1 f, \ldots, R_n f \in L^p_{w_2}$ for some $p > \mu$ and w_2 satisfying $A_{p/\mu}$ $(= A_{pn/(n-1)})$, then $N(F) \in L^p_{w_2}$ and

$$\|N(F)\|_{p,w_2} \leqslant c[\|f\|_{p,w_2} + \sum_{j=1}^{n} \|R_j f\|_{p,w_2}].$$

Finally, as a corollary of Theorem 1, we will show that if f, $R_j f \in L^1$ $(w \equiv 1)$ for all j, then the Fourier transforms satisfy the standard formula

$$(R_j f)^\wedge(x) = i \frac{x_j}{|x|} \hat{f}(x)$$

for $x \neq 0$, and, by continuity, $\widehat{(R_j f)}(0) = \hat{f}(0) = 0$. The simple proof is given at the end of §3.

§2. Preliminary results

In this section, we prove some facts, including Theorem 2, which will be useful later.

First, we need several observations about condition A_1. If g^* denotes the Hardy-Littlewood maximal function of a function g, it is not hard to see that $w \in A_1$ if and only if there is a constant c such that

$$w^*(x) \leqslant c w(x) \text{ a.e.} \tag{4}$$

It is also easy to check that if $w \in A_1$ and I and J are cubes with $I \subset J$, then

$$\int_J w dx \leqslant c \frac{|J|}{|I|} \int_I w dx . \tag{5}$$

Since for any w that is not identically zero, there is a constant $c > 0$ such that $w^*(x) \geqslant c(1+|x|)^{-n}$, we obtain that $w(x) \geqslant c(1+|x|)^{-n}$ a.e. if $w \in A_1$. Actually, if $w \in A_1$, there exists δ, $0 < \delta < 1$, such that $w^{1/\delta} \in A_1$ (see [7]), so that $w(x) \geqslant c(1+|x|)^{-n\delta}$ a.e. This shows that if $f \in L^1_w$, $w \in A_1$, then $Pf(x, t)$ and $Q_j f(x, t)$ are finite and tend to zero as $t \to +\infty$ (for fixed x). In fact, the estimate implies that

$$\sup_y \frac{w(y)^{-1}}{(t+|x-y|)^n} \qquad ((x, t) \quad \text{fixed}, \quad t > 0) \tag{6}$$

is finite and tends to zero as $t \to +\infty$. Thus, since $P(x-y, t)$ and $Q_j(x-y, t)$ are bounded in absolute value by a multiple of $(t+|x-y|)^{-n}$, it follows that $|Pf(x, t)|$ and $|(Q_j f)(x, t)|$ are bounded by

$$c\int_{R^n} |f(y)| \frac{dy}{(t+|x-y|)^n} \leqslant c\,\|f\|_{1,w}\left\{\sup_y \frac{w(y)^{-1}}{(t+|x-y|)^n}\right\},$$

which is finite and tends to zero as $t \to +\infty$.

In addition to the pointwise existence of $R_j f$ for $f \in L^1_w$, $w \in A_1$, there is also a weak-type estimate: if $m_w(E)$ denotes the w-measure of a set E (i.e., $m_w(E) = \int_E w\,dx$) and if $R_j^* f$ is defined by

$$(R_j^* f)(x) = \sup_{\varepsilon>0} |(R_{j,\varepsilon} f)(x)|,$$

then

$$m_w\{x\colon (R_j^* f)(x) > \lambda\} \leqslant c\lambda^{-1} \|f\|_{1,w}, \ \lambda > 0,$$

with c independent of f and λ. A similar estimate holds for f^*. (See [2], [7].)

We need several facts about condition $A_p, p > 1$, all of which can be found in [2], [6], and [7]. Here we note only that if $w \in A_p$, $p > 1$, there is a constant c such that

$$(B_p) \qquad \int_{R^n} \frac{w(y)}{(t+|x-y|)^{np}}\,dy \leqslant ct^{-np} \int_{|x-y|<t} w(y)\,dy, \ t > 0.$$

(Cf. lemma 1 of [6].) In particular, $w(y)/(1+|y|)^{np}$ is integrable over R^n if $w \in A_p$. This shows that Pf and $Q_j f$ are finite if $f \in L^p_w$, $w \in A_p$, $p > 1$. In fact, by Hölder's inequality,

$$\int_{R^n} |f(y)| \frac{dy}{(t+|x-y|)^n} \leqslant \|f\|_{p,w}\left(\int_{R^n} \frac{w(y)^{-p'/p}}{(t+|x-y|)^{np'}}\,dy\right)^{1/p'},$$

$p' = p/(p-1)$. Since $w \in A_p$, we have $w^{-p'/p} \in A_{p'}$, so that

$$w(y)^{-p'/p}/(1+|y|)^{np'}$$

is integrable and the last expression is finite.

We need the following lemma about harmonic majorization.

LEMMA 1. *Let* $s(x,t)$ *be subharmonic in* R^{n+1}_+ *and satisfy*

$$\sup_{t>0} \int_{R^n} |s(x,t)|^p w(x)\,dx < +\infty$$

for some p, $1 \leqslant p < +\infty$, *with* $w \in A_p$. *Then for* $a > 0$,

$$(7) \qquad s(x,t+a) \leqslant P\big(s(\cdot,a)\big)(x,t).$$

If s *is harmonic, equality holds in* (7).

Proof. First note by the remarks above that $P(s(.,a))(x,t)$ is finite, since $s(.,a) \in L^p_w$, $w \in A_p$. Inequality (7) is a corollary of Theorem 2 of [8], provided that we show

$$\text{(a)} \qquad \sup_{t>0} \int_{R^n} \frac{|s(x,t)|}{(1+t+|x|)^{n+1}}\, dx < +\infty ,$$

$$\text{(b)} \qquad \lim_{t\to+\infty} \int_{R^n} \frac{|s(x,t)|}{(1+t+|x|)^{n+1}}\, dx = 0 .$$

If $p > 1$,

$$\int_{R^n} \frac{|s(x,t)|}{(1+t+|x|)^{n+1}}\, dx$$

$$\leqslant \left(\int_{R^n} |s(x,t)|^p w(x)\, dx\right)^{1/p} \left(\int_{R^n} \frac{w(x)^{-p'/p}}{(1+t+|x|)^{(n+1)p'}}\, dx\right)^{1/p'}$$

$$\leqslant c\left(\int_{R^n} \frac{w(x)^{-p'/p}}{(1+t+|x|)^{(n+1)p'}}\, dx\right)^{1/p'} .$$

Since $(1+t+|x|)^{(n+1)p'} \geqslant (1+t)^{p'}(1+|x|)^{np'}$ and $w^{-p'/p}$ satisfies $B_{p'}$, the last expression is at most

$$\frac{c}{1+t}\left(\int_{R^n} \frac{w(x)^{-p'/p}}{(1+|x|)^{np'}}\, dx\right)^{1/p'} \leqslant \frac{c}{1+t}\left(\int_{|x|<1} w(x)^{-p'/p}\, dx\right)^{1/p'},$$

from which (a) and (b) follow. The argument for $p = 1$ is similar, using for example the simple estimate $w(x)^{-1} \leqslant c(1+|x|)^n$. Finally, if s is harmonic then $s(x, t+a) = P(s(.,a))(x,t)$, by applying (7) to both s and $-s$.

Lemma 2. *Let F be a Cauchy-Riemann system for which*

$$\sup_{t>0} \int_{R^n} |F(x,t)|^p w(x)\, dx < +\infty ,$$

where $\dfrac{n-1}{n} < p < \infty$ *and* $w \in A_{pn/(n-1)}$. *Then* $F(x,t)$ *converges a.e. to a limit* $F(x,0)$ *as* $t \to 0$. *Moreover,* $\| F(x,t) - F(x,0) \|_{p,w} \to 0$ *as* $t \to 0$, *and there is a constant* c *depending only on* n *such that*

$$\text{(8)} \qquad N(F)(x) \leqslant c\left(|F(x,0)|^{\frac{n-1}{n}}\right)^{*\,\frac{n}{n-1}},$$

where $*$ *denotes the Hardy-Littlewood maximal function.*

Proof. Except for the last estimate, this lemma is proved in [4]. The method is standard. Let $q = pn/(n-1)$ and $s(y, t) = |F(y, t)|^{\frac{n-1}{n}} = |F(y, t)|^{\frac{p}{q}}$. Then s is non-negative, continuous and (by [9]) subharmonic in R^{n+1}_{+}. Also,

$$\int_{R^n} s(y, t)^q w(y)\, dy = \int_{R^n} |F(y, t)|^p w(y)\, dy \leqslant c_1, \quad t > 0.$$

Since $q > 1$, there exist $\{t_k\} \to 0$ and $h \in L^q_w$ such that $\| h \|^q_{q,w} \leqslant c_1$ and $s(., t_k)$ converges weakly in L^q_w to h—i.e.,

$$\int_{R^n} s(y, t_k)\, g(y)\, w(y)\, dy \to \int_{R^n} h(y)\, g(y)\, w(y)\, dy$$

if $g \in L^{q'}_w$, $q' = q/(q-1)$. For fixed (x, t), choose $g(y) = P(x-y, t)\, w(y)^{-1}$. Since $w \in A_q$, we have $w^{-q'/q}$ $(= w^{1-q'}) \in B_{q'}$, and therefore $g \in L^{q'}_w$. For this g, the integral on the left above equals $P(s(., t_k))(x, t)$, which majorizes $s(x, t+t_k)$ by Lemma 1, and the integral on the right equals $(Ph)(x, t)$. Hence,

$$s(x, t) = \lim_{t_k \to 0} s(x, t+t_k) \leqslant (Ph)(x, t).$$

Therefore, $|F(x, t)| \leqslant (Ph)(x, t)^{\frac{q}{p}}$, so that

$$N(F)(x) \leqslant N(h)(x)^{\frac{q}{p}} \leqslant ch^*(x)^{\frac{q}{p}}. \tag{9}$$

We have

$$\int_{R^n} h^{*q} w\, dx \leqslant c \int_{R^n} |h|^q w\, dx$$

by [7]. Hence, h^*, and so $N(F)$, is finite a.e., and it follows from [1] that F has non-tangential boundary values $F(x, 0)$ a.e. Moreover,

$$\int_{R^n} |F(x, t) - F(x, 0)|^p w(x)\, dx \to 0 \quad \text{as} \quad t \to 0$$

by dominated convergence:

$$|F(x, t)| \leqslant N(F)(x) \leqslant ch^*(x)^{q/p} \in L^p_w.$$

Since $s(x,t) = |F(x,t)|^{\frac{n-1}{n}}$, we have $s(x,t) \to |F(x,0)|^{\frac{n-1}{n}}$ a.e. This convergence is also in L_w^q norm since

$$| s(x,t) | \leqslant N(F)(x)^{\frac{p}{q}} \in L_w^q .$$

Since $s(.,t_k)$ also converges weakly in L_w^q to h, it follows that $h(x) = |F(x,0)|^{\frac{n-1}{n}}$ a.e. Inequality (8) now follows immediately from (9).

Proof of Theorem 2. Let F be a Cauchy-Riemann system satisfying

$$\sup_{t>0} \int_{R^n} | F(x,t) |^p w_1(x)\, dx < +\infty ,$$

where

$$\frac{n-1}{n} < p < \infty , \quad w_1 \in A_{pn/(n-1)} .$$

Then F has boundary value $F(x,0)$ a.e. and in L_w^p by Lemma 2; moreover,

$$N(F)(x) \leqslant c\left(|F(x,0)|^{\frac{n-1}{n}}\right)^{*\frac{n}{n-1}} .$$

If we now assume that

$$| F(x,0) | \in L_{w_2}^r , \quad \frac{n-1}{n} < r < \infty , \quad w_2 \in A_{rn/(n-1)} ,$$

then

$$\int_{R^n} N(F)(x)^r w_2(x)\, dx \leqslant c \int_{R^n} \left(|F(x,0)|^{\frac{n-1}{n}}\right)^{*\frac{nr}{n-1}} w_2(x)\, dx$$

$$\leqslant c \int_{R^n} | F(x,0) |^r w_2(x)\, dx$$

by [7]. This gives (3) immediately.

Remark. We note in passing that if

$$\sup_{t>0} \int_{R^n} | F(x,t) |^p w(x)\, dx < +\infty , \quad \frac{n-1}{n} < p < \infty , \quad w \in A_{pn/(n-1)} ,$$

then

(10) $$\sup_{t>0} \int_{R^n} | F(x,t) |^p w(x)\, dx \approx \| F(\cdot,0) \|_{p,w}^p .$$

This follows from Theorem 2: the right-hand side is at most a multiple of the left since $F(x, t) \to F(x, 0)$ in L^p_w; the converse inequality is just (3) with w_2 and r chosen to be w and p, resp.

§3. Proof of Theorems 1 and 3

We will prove Theorem 1 first, beginning with part (i). Let $F \in H^1_w$, $F = (u, v_1, \ldots, v_n)$, $w \in A_1$. By Theorem 2, F has boundary values $F(x, 0) = (f(x), g_1(x), \ldots, g_n(x))$ pointwise a.e. and in L^1_w. In particular, $f, g_1, \ldots, g_n \in L^1_w$. We will show that $u = P(f)$ and $v_j = P(g_j)$. Since $u(x, s)$ converges to $f(x)$ in L^1_w, $P(u(., s))(x, t) \to (Pf)(x, t)$ as $s \to 0$:

$$| P(u(\cdot, s))(x, t) - (Pf)(x, t) | = | \int_{R^n} [u(y, s) - f(y)] P(x-y, t)\, dy |$$

$$\leqslant \| u(\cdot, s) - f \|_{1,w} \{ \sup_y w(y)^{-1} P(x-y, t) \},$$

where the expression in curly brackets is finite for each (x, t) (see (6)). By Lemma 1, $u(x, s+t) = P(u(., s))(x, t)$ since u is harmonic. Hence, letting $s \to 0$, we obtain $u(x, t) = (Pf)(x, t)$, as desired. The argument proving that $v_j = P(g_j)$ is similar.

Now let $G = (Pf, Q_1 f, \ldots, Q_n f)$. Then G is a Cauchy-Riemann system with the same first component as F. This implies that the first component of F-G is zero, and so that the others are independent of t; that is, $v_j - Q_j f$ is independent of t. Thus, $v_j = Q_j f$ if both $v_j(x, t)$ and $(Q_j f)(x, t)$ tend to zero as $t \to +\infty$ (x fixed). We have already observed this for $Q_j f$. For v_j, the mean-value property of harmonic functions gives

$$| v_j(x, t) | \leqslant c t^{-n-1} \iint_{|\xi - x|^2 + |t-\eta|^2 < t^2} | v_j(\xi, \eta) |\, d\xi d\eta$$

$$\leqslant c t^{-n} \sup_{\eta > 0} \int_{|\xi - x| < t} | v_j(\xi, \eta) |\, d\xi$$

$$\leqslant c t^{-n} \left(\sup_{\eta > 0} \int_{R^n} | v_j(\xi, \eta) |\, w(\xi)\, d\xi \right) \left(\sup_{\xi :\, |\xi - x| < t} w(\xi)^{-1} \right)$$

$$\leqslant c t^{-n} \sup_{\xi : |\xi - x| < t} w(\xi)^{-1}.$$

Since $w(\xi)^{-1} \leqslant c(1+|\xi|)^{n\delta}$ for some δ, $0 < \delta < 1$, we have

$$|v_j(s,t)| \leqslant ct^{-n}(1+|x|+t)^{n\delta}.$$

Hence, $v_j(x,t) \to 0$ as $t \to \infty$ for each x.

We now know $u = Pf$, $v_j = P(g_j) = Q_j f$. Letting $t \to 0$ in the equation $P(g_j)(x,t) = (Q_j f)(x,t)$ gives $g_j(x) = (R_j f)(x)$ a.e. Thus, $R_j f \in L_w^1$ and $v_j = P(R_j f) = Q_j f$, as desired. All that remains to prove in (i) is that $|||F|||$ and $\|f\|_{1,w} + \sum_{j=1}^{n} \|R_j f\|_{1,w}$ are equivalent. This, however, follows immediately from (10) with $p = 1$, since

$$F(x,0) = \big(f(x), R_1 f(x), \ldots, R_n f(x)\big).$$

To prove (ii), let f be a function in L_w^1 for which each $R_j f \in L_w^1$. (The existence of $R_j f$ as a pointwise limit is guaranteed by the hypothesis $w \in A_1$.) We will show that the vector defined by

$$F = (Pf, Q_1 f, \ldots, Q_n f)$$

is in H_w^1. Once this is done, the rest of (ii) clearly follows from (i). We know F is a Cauchy-Riemann system, and only need to show $|||F||| < +\infty$. As $t \to 0$, $F(x,t)$ converges a.e. to $(f, R_1 f, \ldots, R_n f) = F(x,0)$, say, so that $|F(x,0)| \in L_w^1$. Hence, $|||F||| < +\infty$ by Theorem 2 if there exist p and w_1, $\frac{n-1}{n} < p < \infty$, $w_1 \in A_{pn/(n-1)}$, such that

$$\sup_{t>0} \int_{R^n} |F(x,t)|^p w_1(x)\,dx < +\infty. \tag{11}$$

We first claim that if $w \in A_1$, there exists $\alpha > 0$ such that the function

$$w_1(x) = \frac{w(x)}{(1+|x|)^\alpha}$$

also belongs to A_1. Note that $(1+|x|)^{-\beta} \in A_1$ if $0 \leqslant \beta < n$, and that there exists $s > 1$ such that $w^s \in A_1$. Hence, for any cube I, Hölder's inequality gives

$$\frac{1}{|I|}\int_I w_1(x)\,dx \leqslant \left(\frac{1}{|I|}\int_I w(x)^s\,dx\right)^{1/s}\left(\frac{1}{|I|}\int_I (1+|x|)^{-\alpha s'}\,dx\right)^{1/s'},$$

$s' = s/(s-1)$. Choose $\alpha > 0$ so small that $\alpha s' < n$. Then both w^s and $(1+|x|)^{-\alpha s'}$ are in A_1, and

$$\frac{1}{|I|}\int_I w_1(x)\,dx \leqslant c\,(\operatorname*{ess\,inf}_I w^s)^{1/s}(\operatorname*{ess\,inf}_I (1+|x|)^{-\alpha s'})^{1/s'}$$
$$= c\,(\operatorname*{ess\,inf}_I w)\,(\operatorname*{ess\,inf}_I (1+|x|)^{-\alpha})$$
$$\leqslant c\ \operatorname*{ess\,inf}_I w_1\,.$$

This proves the claim.

With this choice of w_1, we will complete the proof of (ii) by showing that (11) holds for any $p < 1$ which is sufficiently close to 1. Let

$$(R^*f)(x) = \max_{j=1,\dots,n} (R_j^*f)(x)\,.$$

Then, as is well-known, there is a constant c depending only on n such that

$$|F(x,t)| \leqslant c\,[f^*(x) + (R^*f)(x)]\,.$$

It follows from the weak-type estimates referred to in §2 that the radial maximal function $N_0(F)(x)\,(=\sup_{t>0}|F(x,t)|)$ satisfies

$$m_w\{x\colon N_0(F)(x) > \lambda\} \leqslant c\lambda^{-1}\,\|f\|_{1,w},\ \lambda > 0\,.$$

We will show that any non-negative function ϕ with

$$m_w\{x\colon \phi(x) > \lambda\} \leqslant c\lambda^{-1}\,,\ \lambda > 0\,,$$

belongs to $L^p_{w_1}$, $1 - \frac{\alpha}{n} < p < 1$. Let $g_r(\lambda)$, $\lambda > 0$, denote the non-increasing rearrangement of a function g with respect to the measure $w(x)\,dx$. Then, by [5], p. 257,

$$\int_{R^n} \phi^p w_1 dx = \int_{R^n} \phi(x)^p (1+|x|)^{-\alpha}\, w(x)\,dx$$
$$\leqslant \int_0^\infty \phi_r^p(\lambda)\,\{(1+|x|)^{-\alpha}\}_r(\lambda)\,d\lambda\,.$$

We have $\phi_r(\lambda) \leqslant c\,\lambda^{-1}$ and must estimate $\{(1+|x|)^{-\alpha}\}_r$. However,

$$m_w\{x\colon (1+|x|)^{-\alpha} > \lambda\} = m_w\{x\colon 1 + |x| < \lambda^{-1/\alpha}\}\,,$$

which for $\lambda \geqslant 1$ is zero and for $0 < \lambda < 1$ is less than

$$\int_{|x|<\lambda^{-1/\alpha}} w dx \leqslant c\lambda^{-n/\alpha} \int_{|x|<1} w\,dx = c\lambda^{-n/\alpha}$$

(see (5)). Therefore,

$$\{(1+|x|)^{-\alpha}\}_r(\lambda) \leqslant c(1+\lambda)^{-\alpha/n}, \ \lambda > 0 .$$

Combining estimates, we obtain

$$\int_{R^n} \phi^p w_1 dx \leqslant c \int_0^\infty \lambda^{-p}(1+\lambda)^{-\alpha/n} d\lambda < +\infty$$

if $1 - \frac{\alpha}{n} < p < 1$, as desired. This completes the proof of (ii).

To prove Theorem 3, let $f \in L^1_w$ and $w \in A_1$. Then (11) holds for F, p and w_1 as in the proof of Theorem 1 (ii). (The proof of (11) does not require $R_j f \in L^1_w$.) Hence, by Lemma 2 (see (8)),

$$N(F)(x) \leqslant c\left(|F(x,0)|^{\frac{n-1}{n}}\right)^{*\frac{n}{n-1}} .$$

Since $F(x,0) = \big(f(x), (R_1 f)(x), \ldots, (R_n f)(x)\big)$, the conclusion of Theorem 3 follows immediately with $\mu = (n-1)/n$.

To prove the fact stated at the end of the introduction, let

$$f, R_1 f, \ldots, R_n f \in L^1 .$$

Clearly,

$$P(R_j f)\hat{}\,(x,t) = \hat{P}(x,t)(R_j f)\hat{}\,(x) = e^{-2\pi t|x|}(R_j f)\hat{}\,(x),$$

$$(Q_j f)\hat{}\,(x,t) = \hat{Q}_j(x,t)\hat{f}(x) = i\,\frac{x_j}{|x|}\,e^{-2\pi t|x|}\hat{f}(x) \text{ a.e.},$$

where the Fourier transform is taken in the x variable with t fixed. (Note that for fixed t, $P(x,t)$ belongs to L^1 and $Q_j(x,t)$ belongs to L^2.) However, these expressions are all equal everywhere since $P(R_j f) = Q_j f$ by Theorem 1 and $P(R_j f) \in L^1$. Therefore, $(R_j f)\hat{}\,(x) = i x_j |x|^{-1} \hat{f}(x)$, as claimed.

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