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This formula is involved in proving that if  $H$  is (algebraically) a subgroup of a Lie group  $G$  and if  $H$  is a closed subset of  $G$ , then  $H$  is a topological Lie subgroup of  $G$  ([3, pp. 96, 105]). Specifically, it implies that  $\{V \text{ in } L(G) \mid \exp(tV) \text{ is in } H, \text{ for all } t \text{ real}\}$  is closed under the bracket. The formula also provides the following geometric interpretation of the bracket  $[X, Y]$  on the Lie algebra  $L(G)$  of a Lie group  $G$ .

COROLLARY 1. If  $X$  and  $Y$  belong to  $L(G)$ , then the curve

$$t \rightarrow \exp(-\sqrt{t}X) \exp(-\sqrt{t}Y) \exp(\sqrt{t}X) \exp(\sqrt{t}Y)$$

has velocity vector  $[X, Y]$  at  $t = 0$ .

#### 4. THE CURVATURE TENSOR

Now assume  $M$  is furnished with an affine connection (covariant differentiation operator)  $\nabla$ .

The *curvature tensor*  $R$  on  $M$  is the  $(^1_3)$ -tensor (equivalently, the linear-transformation-valued bilinear mapping)  $R$  defined by

$$\begin{aligned} R(X, Y)A &= \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X, Y]} A \\ &= ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) A, \end{aligned}$$

for  $X$ ,  $Y$ , and  $A$  vector fields on  $M$ . The relationship between this tensor and the Riemann curvature (in a Riemannian manifold) may be found in [4, pp. 72-73], [2, Chapter 9], and [5, pp. 125-127]. Here we shall show its relationship to parallel translation.

Consider the figure again, and let  $A$  be any vector field on  $M$ . We shall compare parallel translation along  $p_0 \rightarrow p_1 \rightarrow p_4$  with that along  $p_0 \rightarrow p_2 \rightarrow p_3$ . Then, by adding the curve  $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p_3$  defined previously (the dotted curve in the figure), we obtain a closed circuit. We shall need the following.

LEMMA 2. (Taylor's Theorem for parallel translation). Let  $X$  be a vector field defined in a neighborhood of a curve  $\gamma$ , let  $T = \gamma'(0)$ , and for any  $t$  in domain  $(\gamma)$ , let  $\tau_t$  denote parallel translation along  $\gamma$  to  $\gamma(t)$ . Then

$$\tau_0 X(\gamma(t)) - X(\gamma(0)) = \sum_{k=1}^n \frac{t^k}{k!} \nabla_T^k X + O(n+1).$$

*Proof.* Apply the real-variable Taylor's Theorem to the function  $f(t) = \tau_0 X(\gamma(t))$  which has values in a finite dimensional vector space.

$$f'(t) = \lim_{h \rightarrow 0} \frac{\tau_0 X(\gamma(t+h)) - \tau_0 X(\gamma(t))}{h}$$

$$= \tau_0 \lim_{h \rightarrow 0} \frac{\tau_t X(\gamma(t+h)) - X(\gamma(t))}{h} = \tau_0 \nabla_{\gamma'(t)} X.$$

Inductively,  $f^{(n)}(t) = \tau_0 (\nabla_{\gamma'(t)}^n X)$  and  $f^{(n)}(0) = \nabla_T^n X$ .

**THEOREM 2.** Let  $X$ ,  $Y$ , and  $A$  be  $C^\infty$  vector fields on the  $C^\infty$  manifold  $M$  with affine connection  $\nabla$ . Let  $p$  belong to  $M$  and consider parallel translation of  $A_p$  around the closed circuit consisting of (in order) the integral curves of  $-X$ ,  $-Y$ ,  $X$ , and  $Y$  (each parameterized on  $[0, t]$ ,  $t$  small), and (backwards along) the curve  $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$ ,  $0 \leq u \leq t$  (see figure). If  $\Delta A$  is the change in  $A_p$  produced by parallel translation around this circuit, then

$$\Delta A = t^2 R(Y, X) A_p + O(3)$$

and hence

$$\lim_{t \rightarrow 0} \frac{\Delta A}{t^2} = R(Y, X) A_p.$$

*Proof.* The calculation is similar to that for the Lie bracket in Theorem 1, except that we must use parallel translation to compare vectors at different points.  $\tau_i$  denotes parallel translation to  $p_i$  along the arc to  $p_i$  from the location of the tangent vector in question. Elsewhere, subscripts denote point of evaluation, as before. From Lemma 2, we have

$$(6) \quad \tau_1 A_4 - A_1 = t \nabla_Y A_1 + \frac{t^2}{2} \nabla_Y^2 A_1 + O(3)$$

$$(7) \quad \tau_0 A_1 - A_0 = t \nabla_X A_0 + \frac{t^2}{2} \nabla_X^2 A_0 + O(3)$$

$$(8) \quad \tau_2 A_3 - A_2 = t \nabla_X A_2 + \frac{t^2}{2} \nabla_X^2 A_2 + O(3)$$

$$(9) \quad \tau_0 A_2 - A_0 = t \nabla_Y A_0 + \frac{t^2}{2} \nabla_Y^2 A_0 + O(3)$$

Apply  $\tau_0$  to both sides of (6) and (8), obtaining (6') and (8'), respectively. Subtracting (8') and (9) from the sum of (6') and (7), we obtain (via Lemma 2),

$$(10) \quad \tau_0 \tau_1 A_4 - \tau_0 \tau_2 A_3 = t^2 [\nabla_X, \nabla_Y] A_0 + O(3)$$

As before, let  $\beta(u) = \sigma(\sqrt{u})$ ,  $0 \leq u \leq t^2$ . Using  $\beta'(0) = [X, Y]_3$  (from Theorem 1), we may, as in the proof of Lemma 2, show that

$$(11) \quad \tau_3 A_4 - A_3 = t^2 \nabla_{[X,Y]} A_3 + O(4).$$

Now apply  $\tau_4$  to (11) and  $\tau_4 \tau_1$  to (10). Taking the difference of the resulting equations and then applying  $\tau_3$  to both sides, we obtain

$$\begin{aligned} \Delta A &= \tau_3 \tau_4 \tau_1 \tau_0 \tau_2 A_3 - A_3 \\ &= t^2 (\tau_3 \tau_4 \nabla_{[X,Y]} A_3 - \tau_3 \tau_4 \tau_1 [\nabla_X, \nabla_Y] A_0) + O(3) \\ &= t^2 (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y]) A_3 + O(3) = -t^2 R(X, Y) A_p + O(3), \end{aligned}$$

since the change produced by dropping the  $\tau$ 's and switching to  $p_3$  may be absorbed in  $O(3)$ . Thus the theorem follows since  $-R(X, Y) = R(Y, X)$ .

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