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Artikel: THE LIE BRACKET AND THE CURVATURE TENSOR
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$$f_3 - f_2 = tXf_2 + \frac{t^2}{2} X^2 f_2 + O(3) \tag{3}$$

$$f_2 - f_0 = tYf_0 + \frac{t^2}{2} Y^2 f_0 + O(3) \tag{4}$$

Subtracting (3) and (4) from the sum of (1) and (2), and applying Lemma 1 again (up to O (2) only), we obtain

$$f_4 - f_3 = t^2 (XYf - YXf)_0 + \frac{t^3}{2} (X Y^2 f - YX^2 f)_0 + O(3),$$

or

$$f_4 - f_3 = t^2 [X, Y]_0 f + O(3) \tag{5}$$

The meaning of this is that $[X, Y]$ measures the degree to which the circuit $p_3 \to p_2 \to p_0 \to p_1 \to p_4$ fails to be closed. Indeed, if $[X, Y] = 0$, then $p_3 = p_4$ (cf. [1, pp. 134-135]).

If we think of $p = p_3$ as the starting point, and (see figure) define $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$ (so that $p_4 = \sigma(t)$), we may re-express (5) as

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_0 f + O(3) = t^2 [X, Y]_p f + O(3),$$

since switching to p changes $[X, Y] f$ by an amount which is only of order O (1).

3. A Particular Case

As a special case, assume X and Y are left invariant vector fields on a Lie group G, i.e., elements of $L(G)$, the Lie algebra of G; and take p to be e, the identity element of the group. Since, in this context, $X_t(p) = p \exp(tX)$, for p in G, we have

$$\sigma(t) = \exp(-tX) \exp(-tY) \exp(tX) \exp(tY).$$

If we assume $f(e) = 0$, Theorem 1 yields

$$\begin{aligned} & f(\exp(-tX) \exp(-tY) \exp(tX) \exp(tY)) \\ & = t^2 [X, Y]_e f + O(3) \\ & = f(\exp\{t^2 [X, Y] + O(3)\}) \end{aligned}$$

and so

$$\exp(-tX) \exp(-tY) \exp(tX) \exp(tY) = \exp(t^2 [X, Y] + O(3)).$$

This formula is involved in proving that if H is (algebraically) a subgroup of a Lie group G and if H is a closed subset of G, then H is a topological Lie subgroup of G ([3, pp. 96, 105]). Specifically, it implies that $\{ V$ in $L(G) \mid \exp(tV)$ is in H, for all t real $\}$ is closed under the bracket. The formula also provides the following geometric interpretation of the bracket $[X, Y]$ on the Lie algebra $L(G)$ of a Lie group G.

COROLLARY 1. If X and Y belong to $L(G)$, then the curve

$$t \to \exp(-\sqrt{t}X) \exp(-\sqrt{t}Y) \exp(\sqrt{t}X) \exp(\sqrt{t}Y)$$

has velocity vector $[X, Y]$ at $t = 0$.

4. THE CURVATURE TENSOR

Now assume M is furnished with an affine connection (covariant differentiation operator) ∇.

The *curvature tensor* R on M is the $\binom{1}{3}$-tensor (equivalently, the linear-transformation-valued bilinear mapping) R defined by

$$\begin{aligned} R(X, Y) A &= \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X,Y]} A \\ &= \left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) A, \end{aligned}$$

for X, Y, and A vector fields on M. The relationship between this tensor and the Riemann curvature (in a Riemannian manifold) may be found in [4, pp. 72-73], [2, Chapter 9], and [5, pp. 125-127]. Here we shall show its relationship to parallel translation.

Consider the figure again, and let A be any vector field on M. We shall compare parallel translation along $p_0 \to p_1 \to p_4$ with that along $p_0 \to p_2 \to p_3$. Then, by adding the curve $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p_3$ defined previously (the dotted curve in the figure), we obtain a closed circuit. We shall need the following.

LEMMA 2. (Taylor's Theorem for parallel translation). Let X be a vector field defined in a neighborhood of a curve γ, let $T = \gamma'(0)$, and for any t in domain (γ), let τ_t denote parallel translation along γ to $\gamma(t)$. Then

$$\tau_0 X(\gamma(t)) - X(\gamma(0)) = \sum_{k=1}^{n} \frac{t^k}{k!} \nabla_T{}^k X + O(n+1).$$