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# THE LIE BRACKET AND THE CURVATURE TENSOR

by Richard L. FABER

## 1. INTRODUCTION

The purpose of this paper is to present simple, coordinate-free proofs of well-known geometric interpretations (Theorems 1 and 2) of the Lie bracket and curvature tensor (in a  $C^\infty$ -manifold with affine connection  $\nabla$ ). These pertain to the traversal of "parallelogram-like" circuits. The standard demonstrations of these interpretations usually make use of finite Taylor expansions in some special coordinate systems (cf. [1, pp. 135-138] for the Lie bracket; [5, pp. 106-108] for the curvature tensor), or repeated application of the multivariable chain rule (cf. [2, pp. 18-19] and [6, pp. 5-38 to 5-42] for the bracket). Spivak ([6, pp. 5-41]) refers to his proof as "an horrendous, but clever, calculation." An application to Lie group theory is given in Corollary 1.

All functions, curves, and vector fields are  $C^\infty$  on a  $C^\infty$  manifold  $M$ . If  $X$  is a vector field on  $M$ , then an *integral curve* of  $X$  is a curve  $\gamma$  (or  $\gamma_X$ ) satisfying  $\gamma'(t) = X(\gamma(t))$ , for all  $t$  in domain  $(\gamma)$ . If, in addition,  $\gamma(0) = p$ , we say that  $\gamma$  is an integral curve starting at  $p$ . We shall use  $X_t$  to denote the *flow* of  $X$ , so that  $X_t(p) = \gamma(t)$ , where  $\gamma$  is an integral curve of  $X$  starting at  $p$ .

## 2. THE LIE BRACKET

If  $f$  is a function on  $M$ , the following is immediate from applying Taylor's Theorem for functions of a real variable to the composition  $f \cdot \gamma$ , and observing that  $(f \cdot \gamma)^{(k)} = X^k f \cdot \gamma$ . Throughout this paper,  $O(n)$  ( $n$  a positive integer) denotes a quantity for which  $O(n)/t^n$  is bounded for small  $t$ .

LEMMA 1. (Taylor's Theorem for integral curves). If  $\gamma$  is an integral curve of a vector field  $X$  and if  $f$  is a real-valued function defined in a neighborhood of image  $(\gamma)$ , then

$$f(\gamma(t)) - f(\gamma(0)) = \sum_{k=1}^n \frac{t^k}{k!} (X^k f)(\gamma(0)) + O(n+1)$$