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THE LIE BRACKET AND THE CURVATURE TENSOR

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1. INTRODUCTION

The purpose of this paper is to present simple, coordinate-free proofs of well-known geometric interpretations (Theorems 1 and 2) of the Lie bracket and curvature tensor (in a C^∞ -manifold with affine connection ∇). These pertain to the traversal of "parallelogram-like" circuits. The standard demonstrations of these interpretations usually make use of finite Taylor expansions in some special coordinate systems (cf. [1, pp. 135-138] for the Lie bracket; [5, pp. 106-108] for the curvature tensor), or repeated application of the multivariable chain rule (cf. [2, pp. 18-19] and [6, pp. 5-38 to 5-42] for the bracket). Spivak ([6, pp. 5-41]) refers to his proof as "an horrendous, but clever, calculation." An application to Lie group theory is given in Corollary 1.

All functions, curves, and vector fields are C^∞ on a C^∞ manifold M. If X is a vector field on M, then an *integral curve* of X is a curve γ (or γ_X) satisfying $\gamma'(t) = X(\gamma(t))$, for all t in domain (γ). If, in addition, $\gamma(0) = p$, we say that γ is an integral curve starting at p. We shall use X_t to denote the *flow* of X, so that $X_t(p) = \gamma(t)$, where γ is an integral curve of X starting at p.

2. THE LIE BRACKET

If f is a function on M, the following is immediate from applying Taylor's Theorem for functions of a real variable to the composition $f \cdot \gamma$, and observing that $(f \cdot \gamma)^{(k)} = X^k f \cdot \gamma$. Throughout this paper, $O(n)$ (n a positive integer) denotes a quantity for which $O(n) / t^n$ is bounded for small t.

LEMMA 1. (Taylor's Theorem for integral curves). If γ is an integral curve of a vector field X and if f is a real-valued function defined in a neighborhood of image (γ), then

$$f(\gamma(t)) - f(\gamma(0)) = \sum_{k=1}^{n} \frac{t^k}{k!} (X^k f)(\gamma(0)) + O(n+1)$$