

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE LIE BRACKET AND THE CURVATURE TENSOR
Autor: Faber, Richard L.
DOI: <https://doi.org/10.5169/seals-48173>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 21.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

THE LIE BRACKET AND THE CURVATURE TENSOR

by Richard L. FABER

1. INTRODUCTION

The purpose of this paper is to present simple, coordinate-free proofs of well-known geometric interpretations (Theorems 1 and 2) of the Lie bracket and curvature tensor (in a C^∞ -manifold with affine connection ∇). These pertain to the traversal of “parallelogram-like” circuits. The standard demonstrations of these interpretations usually make use of finite Taylor expansions in some special coordinate systems (cf. [1, pp. 135-138] for the Lie bracket; [5, pp. 106-108] for the curvature tensor), or repeated application of the multivariable chain rule (cf. [2, pp. 18-19] and [6, pp. 5-38 to 5-42] for the bracket). Spivak ([6, pp. 5-41]) refers to his proof as “an horrendous, but clever, calculation.” An application to Lie group theory is given in Corollary 1.

All functions, curves, and vector fields are C^∞ on a C^∞ manifold M . If X is a vector field on M , then an *integral curve* of X is a curve γ (or γ_X) satisfying $\gamma'(t) = X(\gamma(t))$, for all t in domain (γ) . If, in addition, $\gamma(0) = p$, we say that γ is an integral curve starting at p . We shall use X_t to denote the *flow* of X , so that $X_t(p) = \gamma(t)$, where γ is an integral curve of X starting at p .

2. THE LIE BRACKET

If f is a function on M , the following is immediate from applying Taylor’s Theorem for functions of a real variable to the composition $f \cdot \gamma$, and observing that $(f \cdot \gamma)^{(k)} = X^k f \cdot \gamma$. Throughout this paper, $O(n)$ (n a positive integer) denotes a quantity for which $O(n) / t^n$ is bounded for small t .

LEMMA 1. (Taylor’s Theorem for integral curves). If γ is an integral curve of a vector field X and if f is a real-valued function defined in a neighborhood of image (γ) , then

$$f(\gamma(t)) - f(\gamma(0)) = \sum_{k=1}^n \frac{t^k}{k!} (X^k f)(\gamma(0)) + O(n+1)$$

THEOREM 1. Let X and Y be C^∞ vector fields on the C^∞ manifold M . Let $p \in M$ and let σ be the curve defined by

$$\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$$

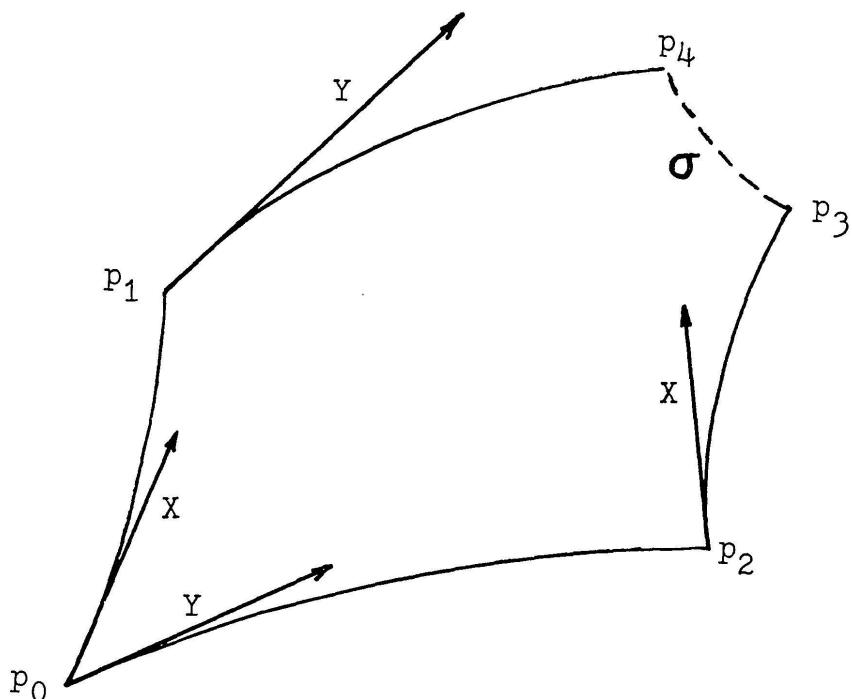
for u sufficiently small. Then for any C^∞ function f on M ,

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_p f + O(3).$$

Accordingly,

$$\lim_{t \rightarrow 0} \frac{f(\sigma(\sqrt{t})) - f(\sigma(0))}{t} = [X, Y]_p f$$

and the curve $\beta(u) = \sigma(\sqrt{u})$ satisfies $\beta'(0) = [X, Y]_p$.



Proof: In the figure, the four solid arcs are integral curves of X or Y , as depicted by the arrows, and all are parameterized on the interval $[0, t]$, for t sufficiently small. E.g., $p_2 = \gamma_X(0)$, $p_3 = \gamma_X(t) = X_t(p_2)$, etc. Subscripts denote the point of evaluation: f_i means $f(p_i)$; Xf_i or $X_i f$ means $(Xf)(p_i)$. The point p in the statement of Theorem 1 coincides with p_3 in the figure. We compute $f_4 - f_1$ by applying Lemma 1 to each arc.

$$(1) \quad f_4 - f_1 = t Y f_1 + \frac{t^2}{2} Y^2 f_1 + O(3)$$

$$(2) \quad f_1 - f_0 = t X f_0 + \frac{t^2}{2} X^2 f_0 + O(3)$$

$$(3) \quad f_3 - f_2 = tXf_2 + \frac{t^2}{2} X^2 f_2 + O(3)$$

$$(4) \quad f_2 - f_0 = tYf_0 + \frac{t^2}{2} Y^2 f_0 + O(3)$$

Subtracting (3) and (4) from the sum of (1) and (2), and applying Lemma 1 again (up to $O(2)$ only), we obtain

$$f_4 - f_3 = t^2 (XYf - YXf)_0 + \frac{t^3}{2} (XY^2f - YX^2f)_0 + O(3),$$

or

$$(5) \quad f_4 - f_3 = t^2 [X, Y]_0 f + O(3)$$

The meaning of this is that $[X, Y]$ measures the degree to which the circuit $p_3 \rightarrow p_2 \rightarrow p_0 \rightarrow p_1 \rightarrow p_4$ fails to be closed. Indeed, if $[X, Y] = 0$, then $p_3 = p_4$ (cf. [1, pp. 134-135]).

If we think of $p = p_3$ as the starting point, and (see figure) define $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$ (so that $p_4 = \sigma(t)$), we may re-express (5) as

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_0 f + O(3) = t^2 [X, Y]_p f + O(3),$$

since switching to p changes $[X, Y] f$ by an amount which is only of order $O(1)$.

3. A PARTICULAR CASE

As a special case, assume X and Y are left invariant vector fields on a Lie group G , i.e., elements of $L(G)$, the Lie algebra of G ; and take p to be e , the identity element of the group. Since, in this context, $X_t(p) = p \exp(tX)$, for p in G , we have

$$\sigma(t) = \exp(-tX) \exp(-tY) \exp(tX) \exp(tY).$$

If we assume $f(e) = 0$, Theorem 1 yields

$$\begin{aligned} & f(\exp(-tX) \exp(-tY) \exp(tX) \exp(tY)) \\ &= t^2 [X, Y]_e f + O(3) \\ &= f(\exp\{t^2 [X, Y] + O(3)\}) \end{aligned}$$

and so

$$\exp(-tX) \exp(-tY) \exp(tX) \exp(tY) = \exp(t^2 [X, Y] + O(3)).$$

This formula is involved in proving that if H is (algebraically) a subgroup of a Lie group G and if H is a closed subset of G , then H is a topological Lie subgroup of G ([3, pp. 96, 105]). Specifically, it implies that $\{V \text{ in } L(G) \mid \exp(tV) \text{ is in } H, \text{ for all } t \text{ real}\}$ is closed under the bracket. The formula also provides the following geometric interpretation of the bracket $[X, Y]$ on the Lie algebra $L(G)$ of a Lie group G .

COROLLARY 1. If X and Y belong to $L(G)$, then the curve

$$t \rightarrow \exp(-\sqrt{t}X) \exp(-\sqrt{t}Y) \exp(\sqrt{t}X) \exp(\sqrt{t}Y)$$

has velocity vector $[X, Y]$ at $t = 0$.

4. THE CURVATURE TENSOR

Now assume M is furnished with an affine connection (covariant differentiation operator) ∇ .

The *curvature tensor* R on M is the $(^1_3)$ -tensor (equivalently, the linear-transformation-valued bilinear mapping) R defined by

$$\begin{aligned} R(X, Y)A &= \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X, Y]} A \\ &= ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) A, \end{aligned}$$

for X , Y , and A vector fields on M . The relationship between this tensor and the Riemann curvature (in a Riemannian manifold) may be found in [4, pp. 72-73], [2, Chapter 9], and [5, pp. 125-127]. Here we shall show its relationship to parallel translation.

Consider the figure again, and let A be any vector field on M . We shall compare parallel translation along $p_0 \rightarrow p_1 \rightarrow p_4$ with that along $p_0 \rightarrow p_2 \rightarrow p_3$. Then, by adding the curve $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p_3$ defined previously (the dotted curve in the figure), we obtain a closed circuit. We shall need the following.

LEMMA 2. (Taylor's Theorem for parallel translation). Let X be a vector field defined in a neighborhood of a curve γ , let $T = \gamma'(0)$, and for any t in domain (γ) , let τ_t denote parallel translation along γ to $\gamma(t)$. Then

$$\tau_0 X(\gamma(t)) - X(\gamma(0)) = \sum_{k=1}^n \frac{t^k}{k!} \nabla_T^k X + O(n+1).$$

Proof. Apply the real-variable Taylor's Theorem to the function $f(t) = \tau_0 X(\gamma(t))$ which has values in a finite dimensional vector space.

$$f'(t) = \lim_{h \rightarrow 0} \frac{\tau_0 X(\gamma(t+h)) - \tau_0 X(\gamma(t))}{h}$$

$$= \tau_0 \lim_{h \rightarrow 0} \frac{\tau_t X(\gamma(t+h)) - X(\gamma(t))}{h} = \tau_0 \nabla_{\gamma'(t)} X.$$

Inductively, $f^{(n)}(t) = \tau_0 (\nabla_{\gamma'(t)}^n X)$ and $f^{(n)}(0) = \nabla_T^n X$.

THEOREM 2. Let X , Y , and A be C^∞ vector fields on the C^∞ manifold M with affine connection ∇ . Let p belong to M and consider parallel translation of A_p around the closed circuit consisting of (in order) the integral curves of $-X$, $-Y$, X , and Y (each parameterized on $[0, t]$, t small), and (backwards along) the curve $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$, $0 \leq u \leq t$ (see figure). If ΔA is the change in A_p produced by parallel translation around this circuit, then

$$\Delta A = t^2 R(Y, X) A_p + O(3)$$

and hence

$$\lim_{t \rightarrow 0} \frac{\Delta A}{t^2} = R(Y, X) A_p.$$

Proof. The calculation is similar to that for the Lie bracket in Theorem 1, except that we must use parallel translation to compare vectors at different points. τ_i denotes parallel translation to p_i along the arc to p_i from the location of the tangent vector in question. Elsewhere, subscripts denote point of evaluation, as before. From Lemma 2, we have

$$(6) \quad \tau_1 A_4 - A_1 = t \nabla_Y A_1 + \frac{t^2}{2} \nabla_Y^2 A_1 + O(3)$$

$$(7) \quad \tau_0 A_1 - A_0 = t \nabla_X A_0 + \frac{t^2}{2} \nabla_X^2 A_0 + O(3)$$

$$(8) \quad \tau_2 A_3 - A_2 = t \nabla_X A_2 + \frac{t^2}{2} \nabla_X^2 A_2 + O(3)$$

$$(9) \quad \tau_0 A_2 - A_0 = t \nabla_Y A_0 + \frac{t^2}{2} \nabla_Y^2 A_0 + O(3)$$

Apply τ_0 to both sides of (6) and (8), obtaining (6') and (8'), respectively. Subtracting (8') and (9) from the sum of (6') and (7), we obtain (via Lemma 2),

$$(10) \quad \tau_0 \tau_1 A_4 - \tau_0 \tau_2 A_3 = t^2 [\nabla_X, \nabla_Y] A_0 + O(3)$$

As before, let $\beta(u) = \sigma(\sqrt{u})$, $0 \leq u \leq t^2$. Using $\beta'(0) = [X, Y]_3$ (from Theorem 1), we may, as in the proof of Lemma 2, show that

$$(11) \quad \tau_3 A_4 - A_3 = t^2 \nabla_{[X,Y]} A_3 + O(4).$$

Now apply τ_4 to (11) and $\tau_4 \tau_1$ to (10). Taking the difference of the resulting equations and then applying τ_3 to both sides, we obtain

$$\begin{aligned} \Delta A &= \tau_3 \tau_4 \tau_1 \tau_0 \tau_2 A_3 - A_3 \\ &= t^2 (\tau_3 \tau_4 \nabla_{[X,Y]} A_3 - \tau_3 \tau_4 \tau_1 [\nabla_X, \nabla_Y] A_0) + O(3) \\ &= t^2 (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y]) A_3 + O(3) = -t^2 R(X, Y) A_p + O(3), \end{aligned}$$

since the change produced by dropping the τ 's and switching to p_3 may be absorbed in $O(3)$. Thus the theorem follows since $-R(X, Y) = R(Y, X)$.

REFERENCES

- [1] BISHOP, R. L. and S. I. GOLDBERG. *Tensor Analysis on Manifolds*. Macmillan Co., New York, 1968.
- [2] — and R. J. CRITTENDEN. *Geometry of Manifolds*. Academic Press, New York, 1964.
- [3] HELGASON, S. *Differential Geometry and Symmetric Spaces*. Academic Press, New York, 1962.
- [4] HICKS, N. J. *Notes on Differential Geometry*. Van Nostrand Co., Princeton, N.J., 1965.
- [5] LAUGWITZ, D. *Differential and Riemannian Geometry*. Academic Press, New York, 1965.
- [6] SPIVAK, M. *Differential Geometry*, Vol. I. M. Spivak, 1970.

(Reçu le 26 juin 1975)

Richard L. Faber

Mathematics Department
Boston College
Chestnut Hill
Massachusetts, 02167