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# THE LIE BRACKET AND THE CURVATURE TENSOR

by Richard L. FABER

## 1. INTRODUCTION

The purpose of this paper is to present simple, coordinate-free proofs of well-known geometric interpretations (Theorems 1 and 2) of the Lie bracket and curvature tensor (in a  $C^\infty$ -manifold with affine connection  $\nabla$ ). These pertain to the traversal of "parallelogram-like" circuits. The standard demonstrations of these interpretations usually make use of finite Taylor expansions in some special coordinate systems (cf. [1, pp. 135-138] for the Lie bracket; [5, pp. 106-108] for the curvature tensor), or repeated application of the multivariable chain rule (cf. [2, pp. 18-19] and [6, pp. 5-38 to 5-42] for the bracket). Spivak ([6, pp. 5-41]) refers to his proof as "an horrendous, but clever, calculation." An application to Lie group theory is given in Corollary 1.

All functions, curves, and vector fields are  $C^\infty$  on a  $C^\infty$  manifold  $M$ . If  $X$  is a vector field on  $M$ , then an *integral curve* of  $X$  is a curve  $\gamma$  (or  $\gamma_X$ ) satisfying  $\gamma'(t) = X(\gamma(t))$ , for all  $t$  in domain  $(\gamma)$ . If, in addition,  $\gamma(0) = p$ , we say that  $\gamma$  is an integral curve starting at  $p$ . We shall use  $X_t$  to denote the *flow* of  $X$ , so that  $X_t(p) = \gamma(t)$ , where  $\gamma$  is an integral curve of  $X$  starting at  $p$ .

## 2. THE LIE BRACKET

If  $f$  is a function on  $M$ , the following is immediate from applying Taylor's Theorem for functions of a real variable to the composition  $f \cdot \gamma$ , and observing that  $(f \cdot \gamma)^{(k)} = X^k f \cdot \gamma$ . Throughout this paper,  $O(n)$  ( $n$  a positive integer) denotes a quantity for which  $O(n)/t^n$  is bounded for small  $t$ .

LEMMA 1. (Taylor's Theorem for integral curves). If  $\gamma$  is an integral curve of a vector field  $X$  and if  $f$  is a real-valued function defined in a neighborhood of image  $(\gamma)$ , then

$$f(\gamma(t)) - f(\gamma(0)) = \sum_{k=1}^n \frac{t^k}{k!} (X^k f)(\gamma(0)) + O(n+1)$$

THEOREM 1. Let  $X$  and  $Y$  be  $C^\infty$  vector fields on the  $C^\infty$  manifold  $M$ . Let  $p \in M$  and let  $\sigma$  be the curve defined by

$$\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$$

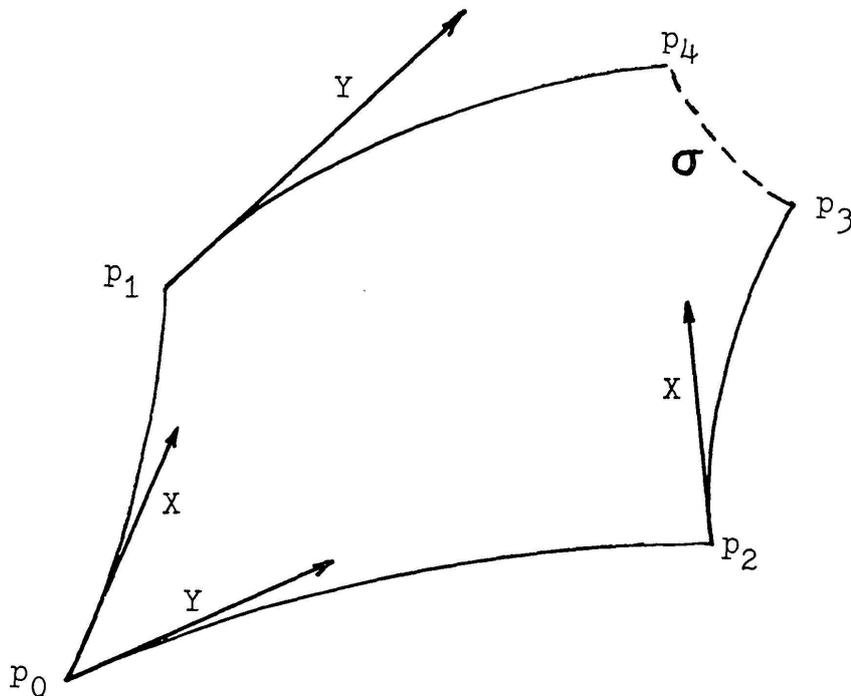
for  $u$  sufficiently small. Then for any  $C^\infty$  function  $f$  on  $M$ ,

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_p f + O(3).$$

Accordingly,

$$\lim_{t \rightarrow 0} \frac{f(\sigma(\sqrt{t})) - f(\sigma(0))}{t} = [X, Y]_p f$$

and the curve  $\beta(u) = \sigma(\sqrt{u})$  satisfies  $\beta'(0) = [X, Y]_p$ .



*Proof:* In the figure, the four solid arcs are integral curves of  $X$  or  $Y$ , as depicted by the arrows, and all are parameterized on the interval  $[0, t]$ , for  $t$  sufficiently small. E.g.,  $p_2 = \gamma_X(0)$ ,  $p_3 = \gamma_X(t) = X_t(p_2)$ , etc. Subscripts denote the point of evaluation:  $f_i$  means  $f(p_i)$ ;  $Xf_i$  or  $X_i f$  means  $(Xf)(p_i)$ . The point  $p$  in the statement of Theorem 1 coincides with  $p_3$  in the figure. We compute  $f_4 - f_3$  by applying Lemma 1 to each arc.

$$(1) \quad f_4 - f_1 = tYf_1 + \frac{t^2}{2} Y^2 f_1 + O(3)$$

$$(2) \quad f_1 - f_0 = tXf_0 + \frac{t^2}{2} X^2 f_0 + O(3)$$

$$(3) \quad f_3 - f_2 = tXf_2 + \frac{t^2}{2} X^2 f_2 + O(3)$$

$$(4) \quad f_2 - f_0 = tYf_0 + \frac{t^2}{2} Y^2 f_0 + O(3)$$

Subtracting (3) and (4) from the sum of (1) and (2), and applying Lemma 1 again (up to  $O(2)$  only), we obtain

$$f_4 - f_3 = t^2 (XYf - YXf)_0 + \frac{t^3}{2} (X Y^2 f - Y X^2 f)_0 + O(3),$$

or

$$(5) \quad f_4 - f_3 = t^2 [X, Y]_0 f + O(3)$$

The meaning of this is that  $[X, Y]$  measures the degree to which the circuit  $p_3 \rightarrow p_2 \rightarrow p_0 \rightarrow p_1 \rightarrow p_4$  fails to be closed. Indeed, if  $[X, Y] = 0$ , then  $p_3 = p_4$  (cf. [1, pp. 134-135]).

If we think of  $p = p_3$  as the starting point, and (see figure) define  $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$  (so that  $p_4 = \sigma(t)$ ), we may re-express (5) as

$$f(\sigma(t)) - f(\sigma(0)) = t^2 [X, Y]_0 f + O(3) = t^2 [X, Y]_p f + O(3),$$

since switching to  $p$  changes  $[X, Y]f$  by an amount which is only of order  $O(1)$ .

### 3. A PARTICULAR CASE

As a special case, assume  $X$  and  $Y$  are left invariant vector fields on a Lie group  $G$ , i.e., elements of  $L(G)$ , the Lie algebra of  $G$ ; and take  $p$  to be  $e$ , the identity element of the group. Since, in this context,  $X_t(p) = p \exp(tX)$ , for  $p$  in  $G$ , we have

$$\sigma(t) = \exp(-tX) \exp(-tY) \exp(tX) \exp(tY).$$

If we assume  $f(e) = 0$ , Theorem 1 yields

$$\begin{aligned} & f(\exp(-tX) \exp(-tY) \exp(tX) \exp(tY)) \\ &= t^2 [X, Y]_e f + O(3) \\ &= f(\exp\{t^2 [X, Y] + O(3)\}) \end{aligned}$$

and so

$$\exp(-tX) \exp(-tY) \exp(tX) \exp(tY) = \exp(t^2 [X, Y] + O(3)).$$

This formula is involved in proving that if  $H$  is (algebraically) a subgroup of a Lie group  $G$  and if  $H$  is a closed subset of  $G$ , then  $H$  is a topological Lie subgroup of  $G$  ([3, pp. 96, 105]). Specifically, it implies that  $\{V \text{ in } L(G) \mid \exp(tV) \text{ is in } H, \text{ for all } t \text{ real}\}$  is closed under the bracket. The formula also provides the following geometric interpretation of the bracket  $[X, Y]$  on the Lie algebra  $L(G)$  of a Lie group  $G$ .

COROLLARY 1. If  $X$  and  $Y$  belong to  $L(G)$ , then the curve

$$t \rightarrow \exp(-\sqrt{t}X) \exp(-\sqrt{t}Y) \exp(\sqrt{t}X) \exp(\sqrt{t}Y)$$

has velocity vector  $[X, Y]$  at  $t = 0$ .

#### 4. THE CURVATURE TENSOR

Now assume  $M$  is furnished with an affine connection (covariant differentiation operator)  $\nabla$ .

The *curvature tensor*  $R$  on  $M$  is the  $\binom{1}{3}$ -tensor (equivalently, the linear-transformation-valued bilinear mapping)  $R$  defined by

$$\begin{aligned} R(X, Y)A &= \nabla_X \nabla_Y A - \nabla_Y \nabla_X A - \nabla_{[X, Y]} A \\ &= ([\nabla_X, \nabla_Y] - \nabla_{[X, Y]}) A, \end{aligned}$$

for  $X, Y$ , and  $A$  vector fields on  $M$ . The relationship between this tensor and the Riemann curvature (in a Riemannian manifold) may be found in [4, pp. 72-73], [2, Chapter 9], and [5, pp. 125-127]. Here we shall show its relationship to parallel translation.

Consider the figure again, and let  $A$  be any vector field on  $M$ . We shall compare parallel translation along  $p_0 \rightarrow p_1 \rightarrow p_4$  with that along  $p_0 \rightarrow p_2 \rightarrow p_3$ . Then, by adding the curve  $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p_3$  defined previously (the dotted curve in the figure), we obtain a closed circuit. We shall need the following.

LEMMA 2. (Taylor's Theorem for parallel translation). Let  $X$  be a vector field defined in a neighborhood of a curve  $\gamma$ , let  $T = \gamma'(0)$ , and for any  $t$  in domain  $(\gamma)$ , let  $\tau_t$  denote parallel translation along  $\gamma$  to  $\gamma(t)$ . Then

$$\tau_0 X(\gamma(t)) - X(\gamma(0)) = \sum_{k=1}^n \frac{t^k}{k!} \nabla_T^k X + O(n+1).$$

*Proof.* Apply the real-variable Taylor's Theorem to the function  $f(t) = \tau_0 X(\gamma(t))$  which has values in a finite dimensional vector space.

$$\begin{aligned} f'(t) &= \lim_{h \rightarrow 0} \frac{\tau_0 X(\gamma(t+h)) - \tau_0 X(\gamma(t))}{h} \\ &= \tau_0 \lim_{h \rightarrow 0} \frac{\tau_t X(\gamma(t+h)) - X(\gamma(t))}{h} = \tau_0 \nabla_{\gamma'(t)} X. \end{aligned}$$

Inductively,  $f^{(n)}(t) = \tau_0 (\nabla_{\gamma'(t)}^n X)$  and  $f^{(n)}(0) = \nabla_T^n X$ .

**THEOREM 2.** Let  $X$ ,  $Y$ , and  $A$  be  $C^\infty$  vector fields on the  $C^\infty$  manifold  $M$  with affine connection  $\nabla$ . Let  $p$  belong to  $M$  and consider parallel translation of  $A_p$  around the closed circuit consisting of (in order) the integral curves of  $-X$ ,  $-Y$ ,  $X$ , and  $Y$  (each parameterized on  $[0, t]$ ,  $t$  small), and (backwards along) the curve  $\sigma(u) = Y_u X_u Y_{-u} X_{-u} p$ ,  $0 \leq u \leq t$  (see figure). If  $\Delta A$  is the change in  $A_p$  produced by parallel translation around this circuit, then

$$\Delta A = t^2 R(Y, X) A_p + O(3)$$

and hence

$$\lim_{t \rightarrow 0} \frac{\Delta A}{t^2} = R(Y, X) A_p.$$

*Proof.* The calculation is similar to that for the Lie bracket in Theorem 1, except that we must use parallel translation to compare vectors at different points.  $\tau_i$  denotes parallel translation to  $p_i$  along the arc to  $p_i$  from the location of the tangent vector in question. Elsewhere, subscripts denote point of evaluation, as before. From Lemma 2, we have

$$(6) \quad \tau_1 A_4 - A_1 = t \nabla_Y A_1 + \frac{t^2}{2} \nabla_Y^2 A_1 + O(3)$$

$$(7) \quad \tau_0 A_1 - A_0 = t \nabla_X A_0 + \frac{t^2}{2} \nabla_X^2 A_0 + O(3)$$

$$(8) \quad \tau_2 A_3 - A_2 = t \nabla_X A_2 + \frac{t^2}{2} \nabla_X^2 A_2 + O(3)$$

$$(9) \quad \tau_0 A_2 - A_0 = t \nabla_Y A_0 + \frac{t^2}{2} \nabla_Y^2 A_0 + O(3)$$

Apply  $\tau_0$  to both sides of (6) and (8), obtaining (6') and (8'), respectively. Subtracting (8') and (9) from the sum of (6') and (7), we obtain (via Lemma 2),

$$(10) \quad \tau_0 \tau_1 A_4 - \tau_0 \tau_2 A_3 = t^2 [\nabla_X, \nabla_Y] A_0 + O(3)$$

As before, let  $\beta(u) = \sigma(\sqrt{u})$ ,  $0 \leq u \leq t^2$ . Using  $\beta'(0) = [X, Y]_3$  (from Theorem 1), we may, as in the proof of Lemma 2, show that

$$(11) \quad \tau_3 A_4 - A_3 = t^2 \nabla_{[X, Y]} A_3 + O(4).$$

Now apply  $\tau_4$  to (11) and  $\tau_4 \tau_1$  to (10). Taking the difference of the resulting equations and then applying  $\tau_3$  to both sides, we obtain

$$\begin{aligned} \Delta A &= \tau_3 \tau_4 \tau_1 \tau_0 \tau_2 A_3 - A_3 \\ &= t^2 (\tau_3 \tau_4 \nabla_{[X, Y]} A_3 - \tau_3 \tau_4 \tau_1 [\nabla_X, \nabla_Y] A_0) + O(3) \\ &= t^2 (\nabla_{[X, Y]} - [\nabla_X, \nabla_Y]) A_3 + O(3) = -t^2 R(X, Y) A_p + O(3), \end{aligned}$$

since the change produced by dropping the  $\tau$ 's and switching to  $p_3$  may be absorbed in  $O(3)$ . Thus the theorem follows since  $-R(X, Y) = R(Y, X)$ .

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