

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE INFINITUDE OF PRIMES; A PROOF USING CONTINUED FRACTIONS
Autor: Barnes, C. W.
Kapitel: 2. Continued Fractions
DOI: <https://doi.org/10.5169/seals-48190>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 05.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

integers x and y . Now suppose the existing primes are p_1, p_2, \dots, p_t with $p_t \geq 5$. Write

$$\sum_{i=2}^t \frac{1}{p_i} = \frac{m}{n}$$

where

$$m = p_2 p_3 \dots p_t + p_1 p_3 \dots p_t + \dots + p_1 p_2 \dots p_{t-1}$$

and $n = p_1 p_2 \dots p_t$. Now $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$ so that $\frac{m}{n} > 1$. Moreover $m > n$, so that $m > 1$ and thus m has a prime factor p_i . This implies

$$p_i \mid p_1 \dots p_{i-1} p_{i+1} \dots p_t$$

and again we have a contradiction.

In the present note we indicate how the theory of simple continued fractions can be used to give a new proof that there exist infinitely many primes. The proof is an application of the theory of periodic continued fractions and the theory of the Pellian equation.

2. CONTINUED FRACTIONS

The necessary material can be found in Perron [3]. We denote the numerators and denominators of the approximants to the simple continued fraction

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

by A_m and B_m respectively for $m = 0, 1, 2, \dots$. Thus $A_0 = a_0$, $A_1 = a_0 a_1 + 1$, $B_0 = 1$, $B_1 = a_1$ and for $m \geq 1$ we have

$$(1) \quad B_{m+1} = a_{m+1} B_m + B_{m-1}.$$

The limit of every infinite periodic simple continued fraction is a quadratic irrational. In particular, if p is a positive integer and

$$x = p + \frac{1}{p + \frac{1}{p + \dots}}$$

then we have

$$(2) \quad x = \frac{p + \sqrt{p^2 + 4}}{2}.$$

Suppose d is a positive integer which is not the square of an integer. The Diophantine equation

$$(3) \quad x^2 - dy^2 = -1$$

is often called the non-Pellian equation. If the simple continued fraction for \sqrt{d} , which is necessarily periodic, has a period consisting of an odd number, m , of terms, then (3) has a solution. In this case every positive solution is of the form $x = A_i, y = B_i$, for $i = qm - 1$ with q odd.

3. THE BASIC RESULT

We use the above results to establish the THEOREM. There exist infinitely many primes.

Proof. Assume that there are only finitely many primes p_1, p_2, \dots, p_t where $p_1 = 2$. Let $p = \prod_{i=1}^t p_i$ and $q = \prod_{i=2}^t p_i$ so that q is the product of the odd primes, and hence $q > 1$. Define x by (2). Then in terms of q we have

$$x = q + \sqrt{q^2 + 1}.$$

Since $q^2 + 1 > 1$ and $p_i \nmid (q^2 + 1)$ for $i = 2, 3, \dots, t$ it follows that $q^2 + 1$ is a power of 2 since 2 is the only remaining prime. Moreover, $q^2 + 1$ must be an odd power of 2 since x is irrational. Thus $q^2 + 1 = 2^{2l+1}$ or

$$q^2 - 2(2^l)^2 = -1$$

and it follows that the non-Pellian equation

$$x^2 - 2y^2 = -1$$

have a solution $x = q, y = 2^l$. Hence $\frac{q}{2^l}$ is an even approximant to the continued fraction for $\sqrt{2}$. We have

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \dots}}$$

and using (1) we easily verify by induction, for this particular continued