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COROLLARY 13.14. If χ is even and real, then $S_{11}(\chi, 2) > 0$ and $S_{21}(\chi, 2) < 0$.

THEOREM 13.15. Let χ be odd. Then

$$S_{11}(\chi, 2) = \frac{iG(\chi)k^2}{\pi} L(1, \bar{\chi}) \tag{13.1}$$

and

$$S_{21}(\chi, 2) = \frac{iG(\chi)k^2}{\pi}\left\{\frac{1}{4}[\bar{\chi}(2)-1]L(1, \bar{\chi}) + \frac{1}{\pi^2}\left[1 - \frac{1}{8}\bar{\chi}(2)\right]L(3, \bar{\chi})\right\}.$$

COROLLARY 13.16. Let χ be odd and real. Then in all cases, $S_{11}(\chi, 2) < 0$; if $\chi(2) = 1$, then $S_{21}(\chi, 2) < 0$.

If χ is real, the class number formula corresponding to (13.1) is due to Cauchy [17]. Pepin [51, p. 205], Lerch [44, p. 395], and Ayoub, Chowla, and Walum [3] have also given proofs of (13.1). Of course, any number of formulas could be proven for $\sum_{a \leq n \leq b} \chi(n) n^r$, where r is a positive integer and a and b are rational multiples of k. However we are unable to make any more non trivial deductions about the positivity (or negativity) of such character sums. In this connection, see [3] and [25].

14. SOME QUESTIONS AND PROBLEMS

In the foregoing work, in order to determine if S_{ji} is of constant sign for classes of real, primitive characters, we expressed S_{ji} as a linear combination of L-functions of real characters evaluated as $s = 1$, and then we inspected the coefficients in this linear combination to determine if all were either non-negative or non-positive. In fact, S_{ji} may always be expressed as a linear combination of L-functions evaluated at $s = 1$. However, in the general situation, the L-functions are associated with complex characters. When non-real characters arise in the representation of S_{ji}, we are unable to say anything about the sign of S_{ji}. We have attempted to find all instances when S_{ji} can be expressed in terms of L-functions of real characters. It is natural to ask if these cases are the only instances when theorems about the non-negativity or non-positivity of S_{ji} are possible. Results of P. D. T. A. Elliott (written communication) appear to indicate that this, indeed, is the case. For example, he has proved the following result. Consider the set of primes p in any residue class, e.g., $p \equiv 1 \pmod 8$, and the as-

sociated characters χ_p of a given fixed order. Then the values of $\arg L(1, \chi_p)$, as p varies, are everywhere dense modulo 2π.

Let us look at just one example where the admittedly scant, numerical evidence seems to suggest otherwise. Let $\chi(n)$ denote the Legendre symbol modulo p, where $p \equiv 1 \pmod 4$. Then S_{51} cannot be expressed in terms of L-functions with real characters. However, for $p \equiv 1 \pmod 8$ and $p \leq 30{,}000$, computations show that $S_{51} > 0$. Sufficient conditions for the positivity of S_{51} are that the two series on the right side of (5.14) are positive. For $p \equiv 1 \pmod 8$, are these two series always positive?

There are a few instances for which we are able to express S_{ji} in terms of L-functions of real characters and for which we are unable to deduce any theorems on the sign of S_{ji}, but for which numerical computations suggest a constant sign. Again, let $\chi(n) = \left(\frac{n}{p}\right)$. For primes p with $p \equiv 7 \pmod 8$ and $p \leq 200{,}000$, calculations of Duncan Buell show that $h(-5p) < \left\{5 - \left(\frac{5}{p}\right)\right\} h(-p)$, or, equivalently, by Corollary 5.3, that $S_{51} > 0$. Is this true for all p with $p \equiv 7 \pmod 8$?

There are 7 additional cases for intervals of length $p/24$ in which numerical calculations for $p \leq 30{,}000$ suggest that $S_{24,i}$ may possibly have a constant sign. For $p \equiv 11 \pmod{24}$, $S_{24,3}$, $S_{24,11} > 0$; for $p \equiv 17 \pmod{24}$, $S_{24,8}$, $S_{24,9} < 0$; for $p \equiv 19 \pmod{24}$, $S_{24,6} > 0$; and for $p \equiv 23 \pmod{24}$, $S_{24,2} = -S_{24,12} > 0$. It can be shown that the above inequalities have the following implications, which we very tenuously conjecture hold for all primes in the given residue classes. If $p \equiv 11 \pmod{12}$, then $h(-12p) < 2h(-8p) + h(-24p)$; if $p \equiv 11 \pmod{24}$, then $h(-8p) < 2h(-p) + h(-12p)$; if $p \equiv 17 \pmod{24}$, then $2h(-3p) < 2h(-8p) + h(-24p)$ and $h(-8p) < 2h(-3p) + h(-4p)$; and if $p \equiv 19 \pmod{24}$, then $4h(-p) < h(-12p) + h(-24p)$.

S. Chowla has conjectured that if p is a prime with $p \equiv 3 \pmod 8$, then S_{21} assumes every value that is a positive, odd multiple of 3. He has also conjectured that if $p \equiv 7 \pmod 8$, then S_{21} assumes every positive, odd integral value. In other words, Chowla has conjectured that $h(-p)$ assumes every possible odd value for each of the sets of primes p with $p \equiv 3 \pmod 8$ and $p \equiv 7 \pmod 8$. Samuel Wagstaff has done some calculations to test Chowla's conjectures and similar conjectures of the author. All of the calculational data are for $p \leq 30{,}000$. For $p \equiv 3 \pmod 8$, the largest value for S_{21} is 297. There are only two omissions, 249 and 291.

For $p \equiv 7 \pmod 8$, the largest value for S_{21} is 259. The smallest value not assumed is 163. There are several other values between 163 and 259 that are not assumed. The calculations also strongly support the following conjectures. S_{41} and S_{31}, for $p \equiv 1 \pmod 4$; S_{52}, for $p \equiv 3 \pmod 4$; S_{81}, for $p \equiv 1 \pmod 8$; S_{82}, for $p \equiv 7 \pmod 8$; $-S_{84}$, for $p \equiv 5 \pmod 8$; and $S_{12,2}$, for $p \equiv 7 \pmod 8$ and for $p \equiv 11 \pmod{12}$, each assumes all positive, integral values. We refer the reader to the foregoing work here for the translations of these conjectures into conjectures about class numbers.

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