Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 22 (1976)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CLASSICAL THEOREMS ON QUADRATIC RESIDUES

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Kapitel: 13. Sums of quadratic residues and nonresidues

DOI: https://doi.org/10.5169/seals-48188

THEOREM 12.5. Let χ be even, and let m be an arbitrary positive integer. Then

$$\sum_{j=0}^{m-1} S_{3m,3j+2} = - \frac{3^{1/2} G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{3k}) .$$

The instances of Theorem 12.5 with $m = 1, 2, 4$ and 8 are consequences of Theorems 4.1, 6.1, 9.1 and 11.1, respectively.

THEOREM 12.6. Let χ be odd, and let m be an arbitrary positive integer. Then

$$S_{5m,2} - S_{5m,4} + S_{5m,7} - S_{5m,9} + - \cdots + S_{5m,5m-3} - S_{5m,5m-1}$$
$$= - \frac{i 5^{1/2} G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{5k}) .$$

The special cases of Theorem 12.6 for $m = 1$ and $m = 2$ follow immediately from Theorems 5.1 and 8.1, respectively.

THEOREM 12.7. Let χ be odd, and let m be an arbitrary positive integer. Then

$$S_{12m,2} + S_{12m,3} + S_{12m,4} + S_{12m,5} - S_{12m,8} - S_{12m,9} - S_{12m,10} - S_{12m,11}$$
$$+ + + + - - - - \cdots - S_{12m,12m-4} - S_{12m,12m-3} - S_{12m,12m-2} - S_{12m,12m-1}$$
$$= - \frac{i (12)^{1/2} G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{12k}) .$$

The special instances of $m = 1$ and $m = 2$ of Theorem 12.7 yield results that are easily deduced from Theorems 9.1 and 11.1, respectively.

The class number formula arising from Theorem 12.1 was first proved by Holden [39]. A less general form of Theorem 12.2 was also established by Holden [36] who in another paper [37] used his result to derive formulas for sums of the Legendre-Jacobi symbol over various residue classes. The special case $m = 1$ of the class number formula deducible from Theorem 12.7 is due to Lerch [44, p. 407]. Otherwise, the results of this section appear to be new.

13. SUMS OF QUADRATIC RESIDUES AND NONRESIDUES

We mentioned in the Introduction the two equivalent formulations of Dirichlet's theorem for primes that are congruent to 3 modulo 4. In this section, we state and prove as many theorems as we can that are of the same nature as (1.3). In the case that $\chi(n)$ is the Legendre symbol, we stated our results in [7, Section 4]. For convenience, we put

$$S_{ji}(\chi, r) = \sum_{(i-1)k/j<n<ik/j} \chi(n)\, n^r,$$

where i, j, and r are natural numbers. Again, χ is primitive throughout the section.

THEOREM 13.1. Let χ be even. Then

$$S_{21}(\chi, 1) = -\frac{G(\chi)\,k}{\pi^2}\left\{1 - \frac{1}{4}\bar{\chi}(2)\right\} L(2, \bar{\chi}).$$

Proof. In (12.1), put $f(x) = x$, $c = 0$, and $d = k/2$. Integrating by parts, we find that

$$S_{21}(\chi, 1) = \frac{G(\chi)\,k}{2\pi^2} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^2} \{\cos(\pi n) - 1\},$$

and the desired result readily follows.

COROLLARY 13.2. For any even, real character χ, we have $S_{21}(\chi, 1) < 0$.

In view of Corollary 3.8 and the fact that $S_{21} = 0$ for even χ, Corollary 13.2 is certainly not surprising.

THEOREM 13.3. Let χ be odd. Then

$$S_{21}(\chi, 1) = \frac{iG(\chi)\,k}{2\pi}\{\bar{\chi}(2) - 1\} L(1, \bar{\chi}).$$

Proof. In (12.2), put $f(x) = x$, $c = 0$, and $d = k/2$. Thus, upon integrating by parts, we get

$$S_{21}(\chi, 1) = \frac{iG(\chi)\,k}{2\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \cos(\pi n),$$

from which the desired result readily follows.

COROLLARY 13.4. If χ is real and odd, then $S_{21}(\chi, 1) > 0$, if $\chi(2) \neq 1$, and $S_{21}(\chi, 1) = 0$, otherwise.

In view of Corollary 3.3 and the elementary fact that $S_{41} = 0$ if $\chi(2) = -1$ [8], at least part of Corollary 13.4 is expected. If p is a prime, Corollary 13.4 shows that the sum of the quadratic residues modulo p exceeds the sum of the non-residues on $(0, p/2)$ if $p \equiv 3 \pmod 8$, while the two sums are equal if $p \equiv 7 \pmod 8$.

THEOREM 13.5. Let χ be odd. Then

$$S_{31}(\chi, 1) = -\frac{iG(\chi)k}{\pi}\left\{\frac{1}{6}[1-\bar{\chi}(3)]\, L(1, \bar{\chi}) + \frac{3^{1/2}}{4\pi} L(2, \bar{\chi}_{3k})\right\}.$$

Proof. In (12.2), put $f(x) = x$, $c = 0$, and $d = k/3$. The result follows from the same type of calculation as above.

COROLLARY 13.6. If χ is real and odd, then $S_{31}(\chi, 1) > 0$.

The following theorems are proved in the same manner as above.

THEOREM 13.7. Let χ be odd. Then

$$S_{32}(\chi, 1) = \frac{iG(\chi)k}{\pi}\left\{\frac{1}{6}[\bar{\chi}(3)-1]\, L(1, \bar{\chi}) + \frac{3^{1/2}}{2\pi} L(2, \bar{\chi}_{3k})\right\}.$$

COROLLARY 13.8. If χ is real and odd and if $\chi(3) = 1$, then $S_{32}(\chi, 1) < 0$.

THEOREM 13.9. Let χ be even. Then

$$S_{42}(\chi, 1) = -\frac{kG(\chi)}{4\pi}\left\{L(1, \bar{\chi}_{4k}) + \frac{1}{\pi}[2-\bar{\chi}(2)]\left[1 - \frac{1}{4}\bar{\chi}(2)\right] L(2, \bar{\chi})\right\}.$$

COROLLARY 13.10. For χ real and even, we have $S_{42}(\chi, 1) < 0$.

THEOREM 13.11. Let χ be odd. Then

$$S_{41}(\chi, 1) = -\frac{iG(\chi)k}{2\pi}\left\{\frac{1}{4}\bar{\chi}(2)[1-\bar{\chi}(2)]\, L(1, \bar{\chi}) + \frac{1}{\pi} L(2, \bar{\chi}_{4k})\right\}$$

and

$$S_{43}(\chi, 1) = \frac{iG(\chi)k}{2\pi}\left\{[\bar{\chi}(2)-1]\left[\frac{3}{4}\bar{\chi}(2) - 1\right] L(1, \bar{\chi}) + \frac{1}{\pi} L(2, \bar{\chi}_{4k})\right\}.$$

COROLLARY 13.12. Let χ be real and odd. If $\chi(2) \neq -1$, then $S_{41}(\chi, 1) > 0$; in any case, $S_{43}(\chi, 1) < 0$.

THEOREM 13.13. Let χ be even. Then

$$S_{11}(\chi, 2) = \frac{G(\chi)k^2}{\pi^2} L(2, \bar{\chi})$$

and

$$S_{21}(\chi, 2) = \frac{G(\chi)k^2}{4\pi^2}\{\bar{\chi}(2) - 2\}\, L(2, \bar{\chi}).$$

Corollary 13.14. If χ is even and real, then $S_{11}(\chi, 2) > 0$ and $S_{21}(\chi, 2) < 0$.

Theorem 13.15. Let χ be odd. Then

$$S_{11}(\chi, 2) = \frac{iG(\chi)k^2}{\pi} L(1, \bar{\chi}) \tag{13.1}$$

and

$$S_{21}(\chi, 2) = \frac{iG(\chi)k^2}{\pi}\left\{\frac{1}{4}[\bar{\chi}(2)-1]\,L(1,\bar{\chi}) + \frac{1}{\pi^2}\left[1 - \frac{1}{8}\bar{\chi}(2)\right]L(3,\bar{\chi})\right\}.$$

Corollary 13.16. Let χ be odd and real. Then in all cases, $S_{11}(\chi, 2) < 0$; if $\chi(2) = 1$, then $S_{21}(\chi, 2) < 0$.

If χ is real, the class number formula corresponding to (13.1) is due to Cauchy [17]. Pepin [51, p. 205], Lerch [44, p. 395], and Ayoub, Chowla, and Walum [3] have also given proofs of (13.1). Of course, any number of formulas could be proven for $\sum_{a \leq n \leq b} \chi(n)\, n^r$, where r is a positive integer and a and b are rational multiples of k. However we are unable to make any more non trivial deductions about the positivity (or negativity) of such character sums. In this connection, see [3] and [25].

14. Some questions and problems

In the foregoing work, in order to determine if S_{ji} is of constant sign for classes of real, primitive characters, we expressed S_{ji} as a linear combination of L-functions of real characters evaluated as $s = 1$, and then we inspected the coefficients in this linear combination to determine if all were either non-negative or non-positive. In fact, S_{ji} may always be expressed as a linear combination of L-functions evaluated at $s = 1$. However, in the general situation, the L-functions are associated with complex characters. When non-real characters arise in the representation of S_{ji}, we are unable to say anything about the sign of S_{ji}. We have attempted to find all instances when S_{ji} can be expressed in terms of L-functions of real characters. It is natural to ask if these cases are the only instances when theorems about the non-negativity or non-positivity of S_{ji} are possible. Results of P. D. T. A. Elliott (written communication) appear to indicate that this, indeed, is the case. For example, he has proved the following result. Consider the set of primes p in any residue class, e.g., $p \equiv 1 \pmod 8$, and the as-