Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CLASSICAL THEOREMS ON QUADRATIC RESIDUES
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Kapitel: 11. SUMS OVER INTERVALS OF LENGTH k/24.
DOI: https://doi.org/10.5169/seals-48188

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and

$$S_{16,4} - S_{16,5} = \frac{G(\chi)}{2\pi i} \{ [2\bar{\chi}(2) - \bar{\chi}(4) - \bar{\chi}(8)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi})$$
$$- 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \} .$$

COROLLARY 10.2. If d is odd and positive, then

$$S_{16,1} + S_{16,8} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$

and

$$S_{16,4} + S_{16,5} > 0 , \quad \text{if} \quad \chi(2) = -1 .$$

If d is odd and negative, then

$$S_{16,2} - S_{16,7} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$
$$S_{16,3} - S_{16,6} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$
$$S_{16,3} - S_{16,6} < 0 , \quad \text{if} \quad \chi(2) = -1 ,$$

and

$$S_{16,4} - S_{16,5} < 0 , \quad \text{if} \quad \chi(2) = 1 .$$

11. SUMS OVER INTERVALS OF LENGTH $k/24$.

For intervals of length $k/24$, a complete statement of Theorem 11.1 for both even and odd characters would require 24 formulas. Because of limitations of space, we state just 2 of the formulas for $S_{24,i}(\chi)$, where $1 \leq i \leq 12$ and χ is even or odd.

THEOREM 11.1. Let χ be even. Let $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, $\chi_{4k}(n) = \chi_4(n) \chi(n)$, $\chi_{8k}(n) = \chi_4(n) \chi_8(n) \chi(n)$, and $\chi_{24k}(n) = \left(\frac{n}{3}\right) \chi_8(n) \chi(n)$.

Then

$$S_{24,1} = \frac{G(\chi)}{2\pi} \left\{ \frac{1}{2} \bar{\chi}(2) [1 + \bar{\chi}(3)] L(1, \bar{\chi}_{4k}) \right.$$
$$+ \frac{1}{4} 3^{1/2} \bar{\chi}(4) [1 + \bar{\chi}(2)] L(1, \bar{\chi}_{3k}) + 2^{-1/2} [\bar{\chi}(3) - 1] L(1, \bar{\chi}_{8k})$$
$$\left. + (3/2)^{1/2} L(1, \bar{\chi}_{24k}) \right\} .$$

Let χ be odd. Put $\chi_{8k}(n) = \chi_8(n)\chi(n)$, $\chi_{12k}(n) = \left(\frac{n}{3}\right)\chi_4(n)\chi(n)$, and $\chi_{24k}(n) = \left(\frac{n}{3}\right)\chi_4(n)\chi_8(n)\chi(n)$. Then

$$S_{24,2} = \frac{G(\chi)}{2\pi i}\left\{\frac{1}{4}\bar{\chi}(2)[2-\bar{\chi}(2)][\bar{\chi}(2)-1][1-\bar{\chi}(3)]L(1,\bar{\chi})\right.$$
$$+ 2^{-1/2}[1+\bar{\chi}(3)]L(1,\bar{\chi}_{8k}) + 3^{1/2}\left[\frac{1}{2}\bar{\chi}(2)-1\right]L(1,\bar{\chi}_{12k})$$
$$\left. + (3/2)^{1/2}L(1,\bar{\chi}_{24k})\right\}.$$

The next result gives the deductions about positive and negative character sums that can be derived from a full statement of Theorem 11.1.

COROLLARY 11.2. If $d > 0$, we have

$S_{24,1} > 0$, if $\chi(2) = \chi(3) = 1$, or if $\chi(2) = 0$ and $\chi(3) = 1$;

$S_{24,3} > 0$, if $\chi(2) = 0$ and $\chi(3) = -1$;

$S_{24,5} > 0$, if $\chi(2) = \chi(3) = -1$;

$S_{24,10} < 0$, if $\chi(2) \neq 1$ and $\chi(3) = -1$, or if $\chi(2) = -1$ and $\chi(3) = 0$;

and

$S_{24,12} < 0$, if $\chi(2) \neq 1$ and $\chi(3) = 1$.

If $d < 0$, we have

$S_{24,4} > 0$, if $\chi(2) = 1$, or if $\chi(2) = 0$ and $\chi(3) = 1$;

$S_{24,6} > 0$, if $\chi(2) = 0$ and $\chi(3) = -1$;

$S_{24,7} > 0$, if $\chi(2) \neq -1$ and $\chi(3) = -1$;

and

$S_{24,9} > 0$, if $\chi(3) = 1$, or if $\chi(2) = -1$ and $\chi(3) = 0$.

We next state just two of the 24 different class number formulas involving $S_{24,i}$ that can be deduced.

COROLLARY 11.3. If $d > 0$, $2 \nmid d$, and $3 \nmid d$, then

$$(11.1)\quad 8S_{24,1}(\chi_d) = \left(\frac{d}{2}\right)\left\{1+\left(\frac{d}{3}\right)\right\}h(-4d) + \left\{1+\left(\frac{d}{2}\right)\right\}h(-3d)$$
$$+\left\{\left(\frac{d}{3}\right)-1\right\}h(-8d) + h(-24d).$$

If $d < 0$, $2 \not| \, d$, and $3 \not| \, d$, then

$$(11.2) \quad 8S_{24,2}(\chi_{-d}) = \left\{2\left(\frac{d}{2}\right) - 1\right\}\left\{\left(\frac{d}{2}\right) - 1\right\}\left\{1 - \left(\frac{d}{3}\right)\right\} h(d) + \left\{1 + \left(\frac{d}{3}\right)\right\} h(8d) + \left\{\left(\frac{d}{2}\right) - 2\right\} h(12d) + h(24d).$$

Several congruences for class numbers may be deduced from Corollary 11.3. We remark that the consideration of other class number formulas involving $S_{24,i}$ does not appear to yield further congruences.

COROLLARY 11.4. If $p \equiv 1 \pmod 4$, then

$$(11.3) \quad h(-24p) + 2h(-4p) + 2h(-3p) \equiv 0 \pmod{16}, \quad \text{if } p \equiv 1 \pmod{48},$$

$$(11.4) \quad h(-24p) + 2h(-4p) + 2h(-3p) \equiv 8 \pmod{16}, \quad \text{if } p \equiv 25 \pmod{48},$$

$$(11.5) \quad h(-24p) - 2h(-8p) \equiv 0 \pmod{16}, \text{ if } p \equiv 5 \pmod{48},$$

$$(11.6) \quad h(-24p) - 2h(-8p) \equiv 8 \pmod{16}, \text{ if } p \equiv 29 \pmod{48},$$

$$(11.7) \quad h(-24p) - 2h(-4p) \equiv 0 \pmod{16}, \text{ if } p \equiv 13 \pmod{48},$$

$$(11.8) \quad h(-24p) - 2h(-4p) \equiv 8 \pmod{16}, \text{ if } p \equiv 37 \pmod{48},$$

$$(11.9) \quad h(-24p) - 2h(-8p) + 2h(-3p) \equiv 0 \pmod{16}, \quad \text{if } p \equiv 17 \pmod{48},$$

and

$$(11.10) \quad h(-24p) - 2h(-8p) + 2h(-3p) \equiv 8 \pmod{16}, \quad \text{if } p \equiv 41 \pmod{48}.$$

Proof. Let $p \equiv j \pmod{48}$, $0 < j < 48$. Then by (11.1), we have

$$(11.11) \quad 8[j/24] \equiv \left(\frac{2}{p}\right)\left\{1 + \left(\frac{3}{p}\right)\right\} h(-4p) + \left\{1 + \left(\frac{2}{p}\right)\right\} h(-3p) + \left\{\left(\frac{3}{p}\right) - 1\right\} h(-8p) + h(-24p) \pmod{16}.$$

Congruences (11.3)-(11.10) now follow directly from (11.11) by considering the eight separate cases modulo 48.

COROLLARY 11.5. We have

$$h(-24p) \equiv 0 \pmod 8, \quad \text{if} \quad p \equiv 1 \pmod{24}, \tag{11.12}$$

and

$$h(-24p) \equiv 4 \pmod 8, \quad \text{if} \quad p \equiv 5, 13, 17 \pmod{24}. \tag{11.13}$$

Proof. Congruence (11.12) is a consequence of (11.3), (11.4), Corollary 3.10, and Corollary 4.4. Secondly, for $p \equiv 5 \pmod{24}$, (11.13) follows from (11.5), (11.6), and Corollary 7.5. Thirdly, for $p \equiv 13 \pmod{24}$, (11.13) follows from (11.7), (11.8), and Corollary 3.10. Lastly, for $p \equiv 17 \pmod{24}$, (11.13) follows from (11.9), (11.10), Corollary 4.4, and Corollary 7.5.

COROLLARY 11.6. If $p > 3$ and $p \equiv 3 \pmod 4$, then

$$h(-24p) - h(-12p) \equiv 0 \pmod{16}, \quad \text{if} \quad p \equiv 7 \pmod{48},$$
$$h(-24p) - h(-12p) \equiv 8 \pmod{16}, \quad \text{if} \quad p \equiv 31 \pmod{48},$$
$$h(-24p) - 3h(-12p) + 2h(-8p) \equiv 0 \pmod{16}, \text{ if } p \equiv 11 \pmod{48}$$
$$h(-24p) - 3h(-12p) + 2h(-8p) \equiv 8 \pmod{16}, \text{ if } p \equiv 35 \pmod{48},$$
$$h(-24p) - 3h(-12p) + 4h(-p) \equiv 0 \pmod{16}, \text{ if } p \equiv 19 \pmod{48},$$
$$h(-24p) - 3h(-12p) + 4h(-p) \equiv 8 \pmod{16}, \text{ if } p \equiv 43 \pmod{48},$$
$$h(-24p) - h(-12p) + 2h(-8p) \equiv 8 \pmod{16}, \text{ if } p \equiv 23 \pmod{48},$$

and

$$h(-24p) - h(-12p) + 2h(-8p) \equiv 0 \pmod{16}, \text{ if } p \equiv 47 \pmod{48}.$$

Proof. Let $p \equiv j \pmod{48}$, $0 < j < 48$. Then (11.2) gives

$$\begin{aligned} 8\{[j/12] - [j/24]\} \\ \equiv \left\{2\left(\frac{2}{p}\right) - 1\right\}\left\{\left(\frac{2}{p}\right) - 1\right\}\left\{1 - \left(\frac{3}{p}\right)\right\} h(-p) + \left\{1 + \left(\frac{3}{p}\right)\right\} h(-8p) \\ + \left\{\left(\frac{2}{p}\right) - 2\right\} h(-12p) + h(-24p) \pmod{16}. \end{aligned} \tag{11.14}$$

All of the desired congruences are immediate consequences of (11.14).

COROLLARY 11.7. We have

$$h(-24p) \equiv 0 \pmod 8, \quad \text{if} \quad p \equiv 11, 19, 23 \pmod{24},$$

and

$$h(-24p) \equiv 4 \pmod 8, \quad \text{if} \quad p \equiv 7 \pmod{24}.$$

Proof. The desired congruences follow from Corollaries 7.5, 9.3, and 11.6.

Lerch [44, pp. 409, 410] has derived some class number formulas in terms of the sums $S_{24,i}$, $1 \leq i \leq 12$. Karpinski [42] and Rédei [57] have also established class number relations of this sort.

12. Sums over several intervals of equal length

In this section, it will be convenient to use the following character analogues of the Poisson summation formula [6, Theorem 2.3], [7, equations (4.1), (4.2)]. Let f be continuous and of bounded variation on $[c, d]$. Let χ be a primitive character of modulus k. If χ is even, then

$$(12.1)\quad \sum_{c \leq n \leq d}{}' \chi(n) f(n) = \frac{2G(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_c^d f(x) \cos(2\pi n x/k)\, dx\,;$$

if χ is odd, then

$$(12.2)\quad \sum_{c \leq n \leq d}{}' \chi(n) f(n) = -\frac{2iG(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_c^d f(x) \sin(2\pi n x/k)\, dx\,.$$

The primes $'$ on the summation signs on the left sides of (12.1) and (12.2) indicate that if c or d is an integer, then the associated summands must be halved.

Throughout the section, it is assumed that χ is a primitive character of modulus k. For each of the theorems below, deductions concerning the signs of the pertinent character sums are trivial. Likewise, the corresponding formulas for class numbers are immediate from (2.4). Thus, none of these obvious corollaries shall be explicitly stated.

Theorem 12.1. Let χ be even, and let m be any positive integer. Then

$$(12.3)\quad S_{4m,1} + S_{4m,4} + S_{4m,5} + S_{4m,8} + S_{4m,9} + \ldots + S_{4m,4m} = \frac{2G(\chi)}{\pi} \bar{\chi}(m)\, L(1, \bar{\chi}_{4k})\,.$$

Proof. Apply (12.1) several times with $f(x) \equiv 1$ in each case and with $(c, d) = (0, k/4m)$, $(3k/4m, 5k/4m)$, $(7k/4m, 9k/4m)$, ..., $((4m-1)\, k/4m, k)$. We then get