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9. SUMS OVER INTERVALS OF LENGTH $k/12$.

THEOREM 8.1. Let χ be even, $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, and $\chi_{4k}(n) = \chi_4(n) \chi(n)$. Then

$$S_{12,1} = \frac{G(\chi)}{2\pi} \left\{ [1 + \bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} \bar{\chi}(2) [1 + \bar{\chi}(2)] L(1, \bar{\chi}_{3k}) \right\},$$

$$S_{12,2} = \frac{G(\chi)}{2\pi} \left\{ -[1 + \bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} [2 + \bar{\chi}(2) - \bar{\chi}(4)] L(1, \bar{\chi}_{3k}) \right\},$$

$$S_{12,3} = \frac{G(\chi)}{2\pi} \left\{ 2L(1, \bar{\chi}_{4k}) - 3^{1/2} [1 + \bar{\chi}(2)] L(1, \bar{\chi}_{3k}) \right\},$$

$$S_{12,4} = \frac{G(\chi)}{2\pi} \left\{ -2L(1, \bar{\chi}_{4k}) + 3^{1/2} L(1, \bar{\chi}_{3k}) \right\},$$

$$S_{12,5} = \frac{G(\chi)}{2\pi} \left\{ [1 + \bar{\chi}(3)] L(1, \bar{\chi}_{4k}) - \frac{1}{2} 3^{1/2} [2 + \bar{\chi}(2) + \bar{\chi}(4)] L(1, \bar{\chi}_{3k}) \right\},$$

and

$$S_{12,6} = \frac{G(\chi)}{2\pi} \left\{ -[1 + \bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} \bar{\chi}(2) [1 + \bar{\chi}(2)] L(1, \bar{\chi}_{3k}) \right\}.$$

Let χ be odd and let $\chi_{12k}(n) = \left(\frac{n}{3}\right) \chi_4(n) \chi(n)$. Then

$$(9.1) \quad S_{12,1} = \frac{G(\chi)}{2\pi i} \left\{ \frac{1}{2} [4 - \bar{\chi}(2) \{ 1 - \bar{\chi}(2) \} \{ 1 - \bar{\chi}(3) \}] L(1, \bar{\chi}) - 3^{1/2} L(1, \bar{\chi}_{12k}) \right\},$$

$$S_{12,2} = \frac{G(\chi)}{2\pi i} \left\{ \left[\frac{1}{2} \bar{\chi}(2) - 1 \right] [1 - \bar{\chi}(2)] [1 - \bar{\chi}(3)] L(1, \bar{\chi}) + 3^{1/2} L(1, \bar{\chi}_{12k}) \right\},$$

$$S_{12,3} = \frac{G(\chi)}{2\pi i} [1 - \bar{\chi}(2)] [1 + \bar{\chi}(2) - \bar{\chi}(3)] L(1, \bar{\chi}),$$

$$S_{12,4} = \frac{G(\chi)}{2\pi i} \{ \bar{\chi}(2) [\bar{\chi}(2) - 1] + 1 - \bar{\chi}(3) \} L(1, \bar{\chi}),$$

$$S_{12,5} = \frac{G(\chi)}{2\pi i} \left\{ \frac{1}{2} [\bar{\chi}(3) - 1] [2 + \bar{\chi}(2) - \bar{\chi}(4)] L(1, \bar{\chi}) + 3^{1/2} L(1, \bar{\chi}_{12k}) \right\},$$

and

$$S_{12,6} = \frac{G(\chi)}{2\pi i} \left\{ \frac{1}{2} [1 - \bar{\chi}(2)] [4 + \bar{\chi}(2) \{ 1 - \bar{\chi}(3) \}] L(1, \bar{\chi}) - 3^{1/2} L(1, \bar{\chi}_{12k}) \right\}.$$

COROLLARY 9.2. If $d > 0$, we have

$$S_{12,1} > 0, \quad \text{if } \chi(2) = 1, \text{ or if } \chi(3) \neq -1;$$

$$S_{12,1} = 0, \quad \text{if } \chi(2) \neq 1 \text{ and } \chi(3) = -1;$$

$$S_{12,2} > 0, \quad \text{if } \chi(2) \neq -1 \text{ and } \chi(3) = -1;$$

$$S_{12,2} = 0, \quad \text{if } \chi(2) = \chi(3) = -1;$$

$$S_{12,2} < 0, \quad \text{if } \chi(2) = -1 \text{ and } \chi(3) \neq -1;$$

$$S_{12,3} > 0, \quad \text{if } \chi(2) = -1;$$

$$S_{12,5} < 0, \quad \text{if } \chi(3) = -1;$$

$$S_{12,6} > 0, \quad \text{if } \chi(2) = 1 \text{ and } \chi(3) = -1;$$

$$S_{12,6} = 0, \quad \text{if } \chi(2) \neq 1 \text{ and } \chi(3) = -1;$$

and

$$S_{12,6} < 0, \quad \text{if } \chi(2) \neq 1 \text{ and } \chi(3) \neq -1.$$

If $d < 0$, we have

$$S_{12,2} > 0, \quad \text{if } \chi(2) = 1, \text{ or if } \chi(3) = 1;$$

- $S_{12,3} > 0$, if $\chi(2) = \chi(3) = -1$, or if $\chi(2) = 0$
and $\chi(3) \neq 1$;
- $S_{12,3} = 0$, if $\chi(2) = 1$, or if $\chi(2) = 0$ and $\chi(3) = 1$,
or if $\chi(2) = -1$ and $\chi(3) = 0$;
- $S_{12,3} < 0$, if $\chi(2) = -1$ and $\chi(3) = 1$;
- $S_{12,4} > 0$, if $\chi(2) = -1$, or if $\chi(2) \neq -1$ and
 $\chi(3) \neq 1$;
- $S_{12,4} = 0$, if $\chi(2) \neq -1$ and $\chi(3) = 1$;
- $S_{12,5} > 0$, if $\chi(2) = -1$, or if $\chi(3) = 1$;

and

$$S_{12,6} < 0, \text{ if } \chi(2) = 1.$$

COROLLARY 9.3. We have

$$h(-12p) \equiv 0 \pmod{8}, \text{ if } p \equiv 23 \pmod{24},$$

and

$$h(-12p) \equiv 4 \pmod{8}, \text{ if } p \equiv 7, 11, 19 \pmod{24}.$$

Proof. From (9.1) and (2.4),

$$(9.2) \quad S_{12,1}(\chi_p) = \frac{1}{4} \left\{ 4 + \left[1 - \left(\frac{2}{p} \right) \right] \left[1 - \left(\frac{3}{p} \right) \right] \right\} h(-p) - \frac{1}{4} h(-12p).$$

If $p \equiv j \pmod{24}$, $1 \leq j \leq 23$, then

$$(9.3) \quad S_{12,1}(\chi_p) \equiv [j/12] \pmod{2}.$$

From (9.2) and (9.3) we deduce that

$$4[j/12] \equiv \left\{ 4 + \left[1 - \left(\frac{2}{p} \right) \right] \left[1 - \left(\frac{3}{p} \right) \right] \right\} h(-p) - h(-12p) \pmod{8}.$$

The desired congruences now readily follow.

The special case, $p \equiv 19 \pmod{24}$, of Corollary 9.3 was important in Stark's work [59]. Brown [13], [14] has also given proofs of this special case.

Some of the class number formulas arising from Theorem 9.1 were actually stated by Gauss [26] with the proofs given by Dedekind [21]. Several class number formulas involving the sums $S_{12,i}$, $1 \leq i \leq 6$, were discovered by Lerch [44, pp. 407, 408, 414], Holden [36], [38], [39], Karpiński [42], and Rédei [57].