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9. Sums over intervals of length $k/12$.

THEOREM 8.1. Let χ be even, $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, and $\chi_{4k}(n) = \chi_4(n) \chi(n)$. Then

$$S_{12,1} = \frac{G(\chi)}{2\pi} \{[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} \bar{\chi}(2)[1+\bar{\chi}(2)] L(1, \bar{\chi}_{3k})\},$$

$$S_{12,2} = \frac{G(\chi)}{2\pi} \{-[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} [2+\bar{\chi}(2)-\bar{\chi}(4)] L(1, \bar{\chi}_{3k})\},$$

$$S_{12,3} = \frac{G(\chi)}{2\pi} \{2L(1, \bar{\chi}_{4k}) - 3^{1/2}[1+\bar{\chi}(2)] L(1, \bar{\chi}_{3k})\},$$

$$S_{12,4} = \frac{G(\chi)}{2\pi} \{-2L(1, \bar{\chi}_{4k}) + 3^{1/2} L(1, \bar{\chi}_{3k})\},$$

$$S_{12,5} = \frac{G(\chi)}{2\pi} \{[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) - \frac{1}{2} 3^{1/2} [2+\bar{\chi}(2)+\bar{\chi}(4)] L(1, \bar{\chi}_{3k})\},$$

and

$$S_{12,6} = \frac{G(\chi)}{2\pi} \{-[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} \bar{\chi}(2)[1+\bar{\chi}(2)] L(1, \bar{\chi}_{3k})\}.$$

Let χ be odd and let $\chi_{12k}(n) = \left(\frac{n}{3}\right) \chi_4(n) \chi(n)$. Then

$$S_{12,1} = \frac{G(\chi)}{2\pi i} \left\{\frac{1}{2} [4-\bar{\chi}(2)\{1-\bar{\chi}(2)\}\{1-\bar{\chi}(3)\}] L(1, \bar{\chi}) - 3^{1/2} L(1, \bar{\chi}_{12k})\right\}, \tag{9.1}$$

$$S_{12,2} = \frac{G(\chi)}{2\pi i} \left\{ \left[\frac{1}{2} \bar{\chi}(2) - 1 \right] [1 - \bar{\chi}(2)] [1 - \bar{\chi}(3)] L(1, \bar{\chi}) + 3^{1/2} L(1, \bar{\chi}_{12k}) \right\},$$

$$S_{12,3} = \frac{G(\chi)}{2\pi i} [1 - \bar{\chi}(2)] [1 + \bar{\chi}(2) - \bar{\chi}(3)] L(1, \bar{\chi}),$$

$$S_{12,4} = \frac{G(\chi)}{2\pi i} \{ \bar{\chi}(2) [\bar{\chi}(2) - 1] + 1 - \bar{\chi}(3) \} L(1, \bar{\chi}),$$

$$S_{12,5} = \frac{G(\chi)}{2\pi i} \left\{ \frac{1}{2} [\bar{\chi}(3) - 1] [2 + \bar{\chi}(2) - \bar{\chi}(4)] L(1, \bar{\chi}) + 3^{1/2} L(1, \bar{\chi}_{12k}) \right\},$$

and

$$S_{12,6} = \frac{G(\chi)}{2\pi i} \left\{ \frac{1}{2} [1 - \bar{\chi}(2)] [4 + \bar{\chi}(2) \{ 1 - \bar{\chi}(3) \}] L(1, \bar{\chi}) - 3^{1/2} L(1, \bar{\chi}_{12k}) \right\}.$$

Corollary 9.2. If $d > 0$, we have

$S_{12,1} > 0$, if $\chi(2) = 1$, or if $\chi(3) \neq -1$;

$S_{12,1} = 0$, if $\chi(2) \neq 1$ and $\chi(3) = -1$;

$S_{12,2} > 0$, if $\chi(2) \neq -1$ and $\chi(3) = -1$;

$S_{12,2} = 0$, if $\chi(2) = \chi(3) = -1$;

$S_{12,2} < 0$, if $\chi(2) = -1$ and $\chi(3) \neq -1$;

$S_{12,3} > 0$, if $\chi(2) = -1$;

$S_{12,5} < 0$, if $\chi(3) = -1$;

$S_{12,6} > 0$, if $\chi(2) = 1$ and $\chi(3) = -1$;

$S_{12,6} = 0$, if $\chi(2) \neq 1$ and $\chi(3) = -1$;

and

$S_{12,6} < 0$, if $\chi(2) \neq 1$ and $\chi(3) \neq -1$.

If $d < 0$, we have

$S_{12,2} > 0$, if $\chi(2) = 1$, or if $\chi(3) = 1$;

$$S_{12,3} > 0\,, \quad \text{if} \quad \chi(2) = \chi(3) = -1\,, \text{ or if } \chi(2) = 0 \text{ and } \chi(3) \neq 1\,;$$

$$S_{12,3} = 0\,, \quad \text{if} \quad \chi(2) = 1\,, \text{ or if } \chi(2) = 0 \text{ and } \chi(3) = 1\,, \text{ or if } \chi(2) = -1 \text{ and } \chi(3) = 0\,;$$

$$S_{12,3} < 0\,, \quad \text{if} \quad \chi(2) = -1 \text{ and } \chi(3) = 1\,;$$

$$S_{12,4} > 0\,, \quad \text{if} \quad \chi(2) = -1\,, \text{ or if } \chi(2) \neq -1 \text{ and } \chi(3) \neq 1\,;$$

$$S_{12,4} = 0\,, \quad \text{if} \quad \chi(2) \neq -1 \text{ and } \chi(3) = 1\,;$$

$$S_{12,5} > 0\,, \quad \text{if} \quad \chi(2) = -1\,, \text{ or if } \chi(3) = 1\,;$$

and

$$S_{12,6} < 0\,, \quad \text{if} \quad \chi(2) = 1\,.$$

Corollary 9.3. We have

$$h(-12p) \equiv 0 \pmod 8\,, \quad \text{if} \quad p \equiv 23 \pmod{24}\,,$$

and

$$h(-12p) \equiv 4 \pmod 8\,, \quad \text{if} \quad p \equiv 7, 11, 19 \pmod{24}\,.$$

Proof. From (9.1) and (2.4),

$$S_{12,1}(\chi_p) = \frac{1}{4}\left\{4 + \left[1 - \left(\frac{2}{p}\right)\right]\left[1 - \left(\frac{3}{p}\right)\right]\right\} h(-p) - \frac{1}{4} h(-12p)\,. \tag{9.2}$$

If $p \equiv j \pmod{24}$, $1 \leqq j \leqq 23$, then

$$S_{12,1}(\chi_p) \equiv [j/12] \pmod 2\,. \tag{9.3}$$

From (9.2) and (9.3) we deduce that

$$4\,[j/12] \equiv \left\{4 + \left[1 - \left(\frac{2}{p}\right)\right]\left[1 - \left(\frac{3}{p}\right)\right]\right\} h(-p) - h(-12p) \pmod 8\,.$$

The desired congruences now readily follow.

The special case, $p \equiv 19 \pmod{24}$, of Corollary 9.3 was important in Stark's work [59]. Brown [13], [14] has also given proofs of this special case.

Some of the class number formulas arising from Theorem 9.1 were actually stated by Gauss [26] with the proofs given by Dedekind [21]. Several class number formulas involving the sums $S_{12,i}$, $1 \leqq i \leqq 6$, were discovered by Lerch [44, pp. 407, 408, 414], Holden [36], [38], [39], Karpinski [42], and Rédei [57].