# 8. Sums over intervals of length h/10.

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The congruences for $p \equiv 3 \pmod 4$ follow from (7.6), (7.7), and (7.10).

COROLLARY 7.6. We have

$$h(-40p) \equiv 0 \pmod 8, \text{ if } p \equiv 1, 9, 31, 39 \pmod{40}$$

and

$$h(-40p) \equiv 4 \pmod 8, \text{ if } p \equiv 11, 19, 21, 29 \pmod{40}.$$

*Proof.* The congruences follow from (5.13) and Corollary 7.5.

The character sums of this section were studied in great detail from an elementary viewpoint by Osborn [50] and Glaisher [27], [28], [29]. Some of the class number formulas in this section can be traced back to Gauss [26] with the proofs given by Dedekind [21]. The formulas

$$\text{(7.11)} \qquad \frac{1}{2}h(-8d) = S_{81}(\chi_d) - S_{84}(\chi_d)$$

and

$$\text{(7.12)} \qquad \frac{1}{2}h(8d) = S_{82}(\chi_{-d}) + S_{83}(\chi_{-d})$$

are due to Dirichlet [23]. Proofs of (7.11) and (7.12) were also given by Lerch [44, pp. 407, 409]. Pepin [51], Hurwitz [40], Glaisher [29], Holden [39], Karpinski [42], and Rédei [57] have also derived class number formulas in terms of $S_{8i}$, $1 \leq i \leq 4$.

For $p \equiv 1 \pmod 8$, Corollary 7.5 was first established by Lerch [45, p. 225]. Brown [14] has proven Corollary 7.5 and all the congruences of Corollary 7.4 involving a single class number. He has also pointed out (personal communication) that the remaining congruences of Corollary 7.4 may be deduced from his work [14] and a paper of Hasse [35]. The latter author [32] has also proved Corollary 7.5 for $p \equiv 7 \pmod 8$. As indicated in the Introduction, Corollaries 7.4 and 7.5 have also been proven by Pizer [52]. The special case of Corollary 7.5 when $p \equiv 19 \pmod{24}$ was brought into prominence by Stark [59]. See also [13].

## 8. SUMS OVER INTERVALS OF LENGTH $k/10$.

As with intervals of length $k/5$, we are able to establish theorems about positive sums for odd $\chi$ only.

THEOREM 8.1. Let $\chi$ be odd and put $\chi_{5k}(n) = \left(\frac{n}{5}\right)\chi(n)$. Then

$$S_{10,1} = \frac{G(\chi)}{4\pi i}\{[4+\{1-\bar{\chi}(2)\}\{\bar{\chi}(5)-1\}]\,L(1,\bar{\chi}) - 5^{1/2}[1+\bar{\chi}(2)]\,L(1,\bar{\chi}_{5k})\},$$

$$S_{10,2} = \frac{G(\chi)}{4\pi i}\{[2-\bar{\chi}(2)][1-\bar{\chi}(5)]\,L(1,\bar{\chi}) + 5^{1/2}\bar{\chi}(2)\,L(1,\bar{\chi}_{5k})\},$$

$$S_{10,3} = \frac{G(\chi)}{4\pi i}\{[2-\bar{\chi}(2)][\bar{\chi}(5)-1]\,L(1,\bar{\chi}) + 5^{1/2}[2+\bar{\chi}(2)]\,L(1,\bar{\chi}_{5k})\},$$

$$S_{10,4} = \frac{G(\chi)}{4\pi i}\{[2-\bar{\chi}(2)][1-\bar{\chi}(5)]\,L(1,\bar{\chi}) - 5^{1/2}\bar{\chi}(2)\,L(1,\bar{\chi}_{5k})\},$$

and

$$S_{10,5} = \frac{G(\chi)}{4\pi i}\{[3-4\bar{\chi}(2)+\bar{\chi}(5)]\,L(1,\bar{\chi}) - 5^{1/2}L(1,\bar{\chi}_{5k})\}.$$

Corollary 8.2. If $d < 0$, we have

$S_{10,1} > 0$, if $\chi(2) = -1$ and $\chi(5) \neq -1$;

$S_{10,1} = 0$, if $\chi(2) = -1$ and $\chi(5) = -1$;

$S_{10,2} > 0$, if $\chi(2) = 1$, or if $\chi(2) = 0$ and $\chi(5) \neq 1$;

$S_{10,2} = 0$, if $\chi(2) = 0$ and $\chi(5) = 1$;

$S_{10,2} < 0$, if $\chi(2) = -1$ and $\chi(5) = 1$;

$S_{10,3} > 0$, if $\chi(5) = 1$;

$S_{10,4} > 0$, if $\chi(2) = -1$, or if $\chi(2) = 0$ and $\chi(5) \neq 1$;

$S_{10,4} = 0$, if $\chi(2) = 0$ and $\chi(5) = 1$;

$S_{10,4} < 0$, if $\chi(2) = \chi(5) = 1$;

and

$S_{10,5} < 0$, if $\chi(2) = 1$.

We shall refrain from writing down any of the class number formulas arising from Theorem 8.1, since no further congruences for class numbers may be deduced. The sums $S_{10,i}$, $1 \leqq i \leqq 5$, appear to have been previously discussed only by Karpinski [42] and by Rédei [57] in connection with class numbers.