

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 22 (1976)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** CLASSICAL THEOREMS ON QUADRATIC RESIDUES  
**Autor:** Berndt, Bruce C.  
**Kapitel:** 7. SUMS OVER INTERVALS OF LENGTH  $k/8$ .  
**DOI:** <https://doi.org/10.5169/seals-48188>

#### Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

#### Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

class number formula for  $S_{61}(\chi_{-d})$  is due to Lerch [44, p. 403], and those for  $S_{62}(\chi_d)$  and  $S_{63}(\chi_d)$  are also due to Lerch [44, p. 414]. In the terminology of class numbers, Holden [36] has established (6.4)-(6.6) in the associated special cases. Some results related to (6.1)-(6.3) were also found by Holden [39].

### 7. SUMS OVER INTERVALS OF LENGTH $k/8$ .

**THEOREM 7.1.** Let  $\chi$  be even, let  $\chi_{4k} = \chi_4\chi$ , and let  $\chi_{8k} = \chi_4\chi_8\chi$ . Then

$$(7.1) \quad S_{81} = \frac{G(\chi)}{2\pi} \left\{ \bar{\chi}(2) L(1, \bar{\chi}_{4k}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{82} = \frac{G(\chi)}{2\pi} \left\{ [2 - \bar{\chi}(2)] L(1, \bar{\chi}_{4k}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{83} = \frac{G(\chi)}{2\pi} \left\{ -[2 + \bar{\chi}(2)] L(1, \bar{\chi}_{4k}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi} \left\{ \bar{\chi}(2) L(1, \bar{\chi}_{4k}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\}.$$

Let  $\chi$  be odd and let  $\chi_{8k} = \chi_8\chi$ . Then

$$(7.2) \quad S_{81} = \frac{G(\chi)}{2\pi i} \left\{ \left[ 2 + \frac{1}{2} \bar{\chi}(4) \{ 1 - \bar{\chi}(2) \} \right] L(1, \bar{\chi}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{82} = \frac{G(\chi)}{2\pi i} \left\{ \bar{\chi}(2) \left[ 1 - \frac{3}{2} \bar{\chi}(2) + \frac{1}{2} \bar{\chi}(4) \right] L(1, \bar{\chi}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{83} = \frac{G(\chi)}{2\pi i} \left\{ \bar{\chi}(2) \left[ -1 + \frac{3}{2} \bar{\chi}(2) - \frac{1}{2} \bar{\chi}(4) \right] L(1, \bar{\chi}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi i} \left\{ \left[ 2 - \frac{1}{2} \bar{\chi}(4) \right] [1 - \bar{\chi}(2)] L(1, \bar{\chi}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\}.$$

We need only prove (7.1) and (7.2), for the remaining formulae can then be deduced from (7.1), (7.2), Theorem 3.2, Theorem 3.7, and elementary considerations. Since the proofs are similar to those in previous sections, we omit them. For the same reasons, proofs in sections 8-11 will not be given.

COROLLARY 7.2. If  $d > 0$ , we have

$$\begin{aligned} S_{81} &> 0, \text{ if } \chi(2) = 1 \text{ or } 0; \\ S_{84} &< 0, \text{ if } \chi(2) = -1 \text{ or } 0; \\ |S_{84}| &< S_{81}, \text{ if } \chi(2) = 1; \\ |S_{81}| &< -S_{84}, \text{ if } \chi(2) = -1; \\ S_{81} &> S_{83}, \text{ if } \chi(2) = 1; \\ S_{82} &> -S_{83}, \text{ if } \chi(2) = -1; \\ S_{82} &< -S_{83}, \text{ if } \chi(2) = 1; \\ S_{82} &> S_{84}, \text{ if } \chi(2) = -1; \\ S_{81} &= S_{83}, \text{ if } \chi(2) = -1; \end{aligned}$$

and

$$S_{82} = S_{84}, \text{ if } \chi(2) = 1.$$

If  $d < 0$ , we have

$$\begin{aligned} S_{82} &> 0, \text{ if } \chi(2) = 1 \text{ or } 0; \\ S_{83} &> 0; \\ S_{84} &< 0, \text{ if } \chi(2) = 1; \\ |S_{82}| &< S_{83}, \text{ if } \chi(2) = -1; \\ S_{81} &> -S_{83}; \\ S_{81} &= -S_{82} = S_{84}, \text{ if } \chi(2) = -1; \end{aligned}$$

and

$$S_{82} = S_{83} = -S_{84}, \text{ if } \chi(2) = 1.$$

Theorem 7.1 yields 8 formulae for class numbers. We shall list just those that we need to derive congruences.

COROLLARY 7.3. Let  $d$  be odd. If  $d > 0$ , then

$$(7.3) \quad S_{81}(\chi_d) = \frac{1}{4} \left( \frac{d}{2} \right) h(-4d) + \frac{1}{4} h(-8d)$$

and

$$(7.4) \quad S_{84}(\chi_d) = \frac{1}{4} \left( \frac{d}{2} \right) h(-4d) - \frac{1}{4} h(-8d).$$

If  $d < 0$ , then

$$(7.5) \quad S_{81}(\chi_{-d}) = \frac{1}{4} \left\{ 5 - \left( \frac{d}{2} \right) \right\} h(d) - \frac{1}{4} h(8d),$$

$$(7.6) \quad S_{83}(\chi_{-d}) = \frac{3}{4} \left\{ 1 - \left( \frac{d}{2} \right) \right\} h(d) + \frac{1}{4} h(8d),$$

and

$$(7.7) \quad S_{84}(\chi_{-d}) = \frac{3}{4} \left\{ 1 - \left( \frac{d}{2} \right) \right\} h(d) - \frac{1}{4} h(8d).$$

COROLLARY 7.4. We have

$$h(-8p) \equiv h(-4p) \pmod{8}, \text{ if } p \equiv 1, 5 \pmod{16},$$

$$h(-8p) \equiv 4 + h(-4p) \pmod{8}, \text{ if } p \equiv 9, 13 \pmod{16},$$

$$h(-8p) \equiv 0 \pmod{8}, \text{ if } p \equiv 15 \pmod{16},$$

$$h(-8p) \equiv 4 \pmod{8}, \text{ if } p \equiv 7 \pmod{16},$$

$$h(-8p) \equiv 2h(-p) \pmod{8}, \text{ if } p \equiv 11 \pmod{16},$$

and

$$h(-8p) \equiv -2h(-p) \pmod{8}, \text{ if } p \equiv 3 \pmod{16}.$$

*Proof.* If  $p \equiv j \pmod{16}$ ,  $1 \leq j \leq 15$ , then

$$(7.8) \quad S_{81} \equiv [j/8] \pmod{2}.$$

Let  $p \equiv 1 \pmod{4}$ . Then the first two congruences follow from (7.3), (7.8), and Corollary 3.10. Let  $p \equiv 3 \pmod{4}$ . Then the latter four congruences follow from (7.5), (7.8), and the fact that  $h(-p)$  is odd.

COROLLARY 7.5. We have

$$h(-8p) \equiv 0 \pmod{4}, \text{ if } p \equiv 1, 7 \pmod{8}$$

and

$$h(-8p) \equiv 2 \pmod{4}, \text{ if } p \equiv 3, 5 \pmod{8}.$$

*Proof.* Let  $p \equiv 1 \pmod{4}$ , and suppose that  $p \equiv j \pmod{16}$ ,  $1 \leq j \leq 15$ .

Then

$$(7.9) \quad S_{81} - S_{84} \equiv [j/8] - [j/2] + [3j/8] \pmod{2}.$$

The congruences for  $p \equiv 1 \pmod{4}$  follow from (7.3), (7.4), and (7.9).

Let  $p \equiv 3 \pmod{4}$ , and suppose that  $p \equiv j \pmod{8}$ ,  $1 \leq j \leq 7$ . Then

$$(7.10) \quad S_{83} - S_{84} \equiv -[j/2] - [j/4] \pmod{2}.$$

The congruences for  $p \equiv 3 \pmod{4}$  follow from (7.6), (7.7), and (7.10).

COROLLARY 7.6. We have

$$h(-40p) \equiv 0 \pmod{8}, \text{ if } p \equiv 1, 9, 31, 39 \pmod{40}$$

and

$$h(-40p) \equiv 4 \pmod{8}, \text{ if } p \equiv 11, 19, 21, 29 \pmod{40}.$$

*Proof.* The congruences follow from (5.13) and Corollary 7.5.

The character sums of this section were studied in great detail from an elementary viewpoint by Osborn [50] and Glaisher [27], [28], [29]. Some of the class number formulas in this section can be traced back to Gauss [26] with the proofs given by Dedekind [21]. The formulas

$$(7.11) \quad \frac{1}{2} h(-8d) = S_{81}(\chi_d) - S_{84}(\chi_d)$$

and

$$(7.12) \quad \frac{1}{2} h(8d) = S_{82}(\chi_{-d}) + S_{83}(\chi_{-d})$$

are due to Dirichlet [23]. Proofs of (7.11) and (7.12) were also given by Lerch [44, pp. 407, 409]. Pepin [51], Hurwitz [40], Glaisher [29], Holden [39], Karpinski [42], and Rédei [57] have also derived class number formulas in terms of  $S_{8i}$ ,  $1 \leq i \leq 4$ .

For  $p \equiv 1 \pmod{8}$ , Corollary 7.5 was first established by Lerch [45, p. 225]. Brown [14] has proven Corollary 7.5 and all the congruences of Corollary 7.4 involving a single class number. He has also pointed out (personal communication) that the remaining congruences of Corollary 7.4 may be deduced from his work [14] and a paper of Hasse [35]. The latter author [32] has also proved Corollary 7.5 for  $p \equiv 7 \pmod{8}$ . As indicated in the Introduction, Corollaries 7.4 and 7.5 have also been proven by Pizer [52]. The special case of Corollary 7.5 when  $p \equiv 19 \pmod{24}$  was brought into prominence by Stark [59]. See also [13].

## 8. SUMS OVER INTERVALS OF LENGTH $k/10$ .

As with intervals of length  $k/5$ , we are able to establish theorems about positive sums for odd  $\chi$  only.

**THEOREM 8.1.** Let  $\chi$  be odd and put  $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$ . Then