

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CLASSICAL THEOREMS ON QUADRATIC RESIDUES
Autor: Berndt, Bruce C.
Kapitel: 6. Sums over intervals of length $k/6$.
DOI: <https://doi.org/10.5169/seals-48188>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Siehe Rechtliche Hinweise.

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. Voir Informations légales.

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. See Legal notice.

Download PDF: 25.05.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

the series on the right side of (5.14) may be written in terms of L -functions of quartic characters. Thus, we are unable to derive any positivity results for character sums.

6. SUMS OVER INTERVALS OF LENGTH $k/6$.

THEOREM 6.1. Let χ be even and let $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$. Then

$$(6.1) \quad S_{61} = \frac{3^{1/2} G(\chi)}{2\pi} \{ 1 + \bar{\chi}(2) \} L(1, \bar{\chi}_{3k}),$$

$$(6.2) \quad S_{62} = - \frac{3^{1/2} G(\chi)}{2\pi} \bar{\chi}(2) L(1, \bar{\chi}_{3k}),$$

and

$$(6.3) \quad S_{63} = - \frac{3^{1/2} G(\chi)}{2\pi} L(1, \bar{\chi}_{3k}).$$

Let χ be odd. Then

$$(6.4) \quad S_{61} = \frac{G(\chi)}{2\pi i} \{ 1 + \bar{\chi}(2) + \bar{\chi}(3) - \bar{\chi}(6) \} L(1, \bar{\chi}),$$

$$(6.5) \quad S_{62} = \frac{G(\chi)}{2\pi i} \{ 2 - \bar{\chi}(2) - 2\bar{\chi}(3) + \bar{\chi}(6) \} L(1, \bar{\chi}),$$

and

$$(6.6) \quad S_{63} = \frac{G(\chi)}{2\pi i} \{ 1 - 2\bar{\chi}(2) + \bar{\chi}(3) \} L(1, \bar{\chi}).$$

We shall not give a proof of Theorem 6.1, because all of the formulas may be deduced from Theorems 3.2 and 4.1 and elementary considerations.

COROLLARY 6.2. If $d > 0$, we have

$S_{61} > 0$, if d is even, or if $\chi(2) = 1$;

$S_{61} = 0$, if $\chi(2) = -1$;

$S_{62} > 0$, if $\chi(2) = -1$;

$S_{62} = 0$, if d is even;

$S_{62} < 0$, if $\chi(2) = 1$;

$S_{63} < 0$, for all d ;

$S_{61} = -S_{63}$, if d is even;

$S_{61} = -2S_{62} = -2S_{63}$, if $\chi(2) = 1$;

and

$S_{62} = -S_{63}$, if $\chi(2) = -1$.

If $d < 0$, we have

$S_{61} > 0$, if d is even and $\chi(3) = 1$ or 0 , or if $\chi(2) = 1$,
or if $\chi(2) = -\chi(3) = -1$;

$S_{61} = 0$, if d is even and $\chi(3) = -1$, or if $\chi(3) = 0$
and $\chi(2) = -1$;

$S_{61} < 0$, if $\chi(2) = \chi(3) = -1$;

$S_{62} > 0$, if d is even and $\chi(3) = -1$, or if $\chi(3) \neq 1$;

$S_{62} = 0$, if $\chi(3) = 1$;

$S_{63} > 0$, if d is even and $\chi(3) \neq -1$, or if $\chi(2) = -1$;

$S_{63} = 0$, if d is even and $\chi(3) = -1$, or if $\chi(2) = \chi(3) = 1$;

and

$S_{63} < 0$, if $\chi(2) = 1$ and $\chi(3) \neq 1$.

We remark here that the results $S_{6i} = 0$, $i = 1, 2, 3$, in Corollary 6.2 may be proven in a completely elementary manner. As an illustration, we prove that $S_{61} = 0$ if χ is even and $\chi(2) = -1$. (The following argument was supplied to the author by Thomas Cusick, Ronald J. Evans, and the author's students in a graduate course in number theory.) Since χ is even and $\chi(2) = -1$, we have

$$\begin{aligned} \sum_{k/3 < n < k/2} \chi(n) &= \sum_{\substack{k/3 < n < k/2 \\ n \text{ even}}} \chi(n) + \sum_{\substack{k/3 < n < k/2 \\ n \text{ odd}}} \chi(n) \\ &= \chi(2) \sum_{k/6 < n < k/4} \chi(n) + \sum_{k/4 < n < k/3} \chi(k-2n) \\ &= - \sum_{k/6 < n < k/3} \chi(n). \end{aligned}$$

As $S_{21} = 0$, it follows from the above that $S_{61} = 0$.

In the case that $\chi(n)$ is the Legendre symbol, the equalities of Corollary 6.2 were derived by Johnson and Mitchell [41].

Of course, using (2.4), we may convert (6.1)-(6.6) into formulas involving class numbers. Since no new, additional congruences for class numbers may be derived from these formulas, we shall not write them down. The

class number formula for $S_{61}(\chi_{-d})$ is due to Lerch [44, p. 403], and those for $S_{62}(\chi_d)$ and $S_{63}(\chi_d)$ are also due to Lerch [44, p. 414]. In the terminology of class numbers, Holden [36] has established (6.4)-(6.6) in the associated special cases. Some results related to (6.1)-(6.3) were also found by Holden [39].

7. SUMS OVER INTERVALS OF LENGTH $k/8$.

THEOREM 7.1. Let χ be even, let $\chi_{4k} = \chi_4\chi$, and let $\chi_{8k} = \chi_4\chi_8\chi$. Then

$$(7.1) \quad S_{81} = \frac{G(\chi)}{2\pi} \left\{ \bar{\chi}(2) L(1, \bar{\chi}_{4k}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{82} = \frac{G(\chi)}{2\pi} \left\{ [2 - \bar{\chi}(2)] L(1, \bar{\chi}_{4k}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{83} = \frac{G(\chi)}{2\pi} \left\{ -[2 + \bar{\chi}(2)] L(1, \bar{\chi}_{4k}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi} \left\{ \bar{\chi}(2) L(1, \bar{\chi}_{4k}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\}.$$

Let χ be odd and let $\chi_{8k} = \chi_8\chi$. Then

$$(7.2) \quad S_{81} = \frac{G(\chi)}{2\pi i} \left\{ \left[2 + \frac{1}{2} \bar{\chi}(4) \{ 1 - \bar{\chi}(2) \} \right] L(1, \bar{\chi}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{82} = \frac{G(\chi)}{2\pi i} \left\{ \bar{\chi}(2) \left[1 - \frac{3}{2} \bar{\chi}(2) + \frac{1}{2} \bar{\chi}(4) \right] L(1, \bar{\chi}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{83} = \frac{G(\chi)}{2\pi i} \left\{ \bar{\chi}(2) \left[-1 + \frac{3}{2} \bar{\chi}(2) - \frac{1}{2} \bar{\chi}(4) \right] L(1, \bar{\chi}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi i} \left\{ \left[2 - \frac{1}{2} \bar{\chi}(4) \right] [1 - \bar{\chi}(2)] L(1, \bar{\chi}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\}.$$

We need only prove (7.1) and (7.2), for the remaining formulae can then be deduced from (7.1), (7.2), Theorem 3.2, Theorem 3.7, and elementary considerations. Since the proofs are similar to those in previous sections, we omit them. For the same reasons, proofs in sections 8-11 will not be given.