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the series on the right side of (5.14) may be written in terms of L-functions of quartic characters. Thus, we are unable to derive any positivity results for character sums.

6. Sums over intervals of length $k/6$.

Theorem 6.1. Let χ be even and let $\chi_{3k}(n) = \left(\frac{n}{3}\right)\chi(n)$. Then

$$S_{61} = \frac{3^{1/2}G(\chi)}{2\pi}\{1 + \bar{\chi}(2)\}L(1, \bar{\chi}_{3k}), \tag{6.1}$$

$$S_{62} = -\frac{3^{1/2}G(\chi)}{2\pi}\bar{\chi}(2)L(1, \bar{\chi}_{3k}), \tag{6.2}$$

and

$$S_{63} = -\frac{3^{1/2}G(\chi)}{2\pi}L(1, \bar{\chi}_{3k}). \tag{6.3}$$

Let χ be odd. Then

$$S_{61} = \frac{G(\chi)}{2\pi i}\{1 + \bar{\chi}(2) + \bar{\chi}(3) - \bar{\chi}(6)\}L(1, \bar{\chi}), \tag{6.4}$$

$$S_{62} = \frac{G(\chi)}{2\pi i}\{2 - \bar{\chi}(2) - 2\bar{\chi}(3) + \bar{\chi}(6)\}L(1, \bar{\chi}), \tag{6.5}$$

and

$$S_{63} = \frac{G(\chi)}{2\pi i}\{1 - 2\bar{\chi}(2) + \bar{\chi}(3)\}L(1, \bar{\chi}). \tag{6.6}$$

We shall not give a proof of Theorem 6.1, because all of the formulas may be deduced from Theorems 3.2 and 4.1 and elementary considerations.

Corollary 6.2. If $d > 0$, we have

$S_{61} > 0$, if d is even, or if $\chi(2) = 1$;

$S_{61} = 0$, if $\chi(2) = -1$;

$S_{62} > 0$, if $\chi(2) = -1$;

$S_{62} = 0$, if d is even;

$S_{62} < 0$, if $\chi(2) = 1$;

$S_{63} < 0$, for all d;

$$S_{61} = -S_{63}, \text{ if } d \text{ is even};$$

$$S_{61} = -2S_{62} = -2S_{63}, \text{ if } \chi(2) = 1;$$

and

$$S_{62} = -S_{63}, \text{ if } \chi(2) = -1.$$

If $d < 0$, we have

$S_{61} > 0$, if d is even and $\chi(3) = 1$ or 0, or if $\chi(2) = 1$, or if $\chi(2) = -\chi(3) = -1$;

$S_{61} = 0$, if d is even and $\chi(3) = -1$, or if $\chi(3) = 0$ and $\chi(2) = -1$;

$S_{61} < 0$, if $\chi(2) = \chi(3) = -1$;

$S_{62} > 0$, if d is even and $\chi(3) = -1$, or if $\chi(3) \neq 1$;

$S_{62} = 0$, if $\chi(3) = 1$;

$S_{63} > 0$, if d is even and $\chi(3) \neq -1$, or if $\chi(2) = -1$;

$S_{63} = 0$, if d is even and $\chi(3) = -1$, or if $\chi(2) = \chi(3) = 1$;

and

$S_{63} < 0$, if $\chi(2) = 1$ and $\chi(3) \neq 1$.

We remark here that the results $S_{6i} = 0$, $i = 1, 2, 3$, in Corollary 6.2 may be proven in a completely elementary manner. As an illustration, we prove that $S_{61} = 0$ if χ is even and $\chi(2) = -1$. (The following argument was supplied to the author by Thomas Cusick, Ronald J. Evans, and the author's students in a graduate course in number theory.) Since χ is even and $\chi(2) = -1$, we have

$$\begin{aligned}\sum_{k/3<n<k/2} \chi(n) &= \sum_{\substack{k/3<n<k/2 \\ n \text{ even}}} \chi(n) + \sum_{\substack{k/3<n<k/2 \\ n \text{ odd}}} \chi(n) \\ &= \chi(2) \sum_{k/6<n<k/4} \chi(n) + \sum_{k/4<n<k/3} \chi(k-2n) \\ &= -\sum_{k/6<n<k/3} \chi(n).\end{aligned}$$

As $S_{21} = 0$, it follows from the above that $S_{61} = 0$.

In the case that $\chi(n)$ is the Legendre symbol, the equalities of Corollary 6.2 were derived by Johnson and Mitchell [41].

Of course, using (2.4), we may convert (6.1)-(6.6) into formulas involving class numbers. Since no new, additional congruences for class numbers may be derived from these formulas, we shall not write them down. The class number formula for $S_{61}(\chi_{-d})$ is due to Lerch [44, p. 403], and those for $S_{62}(\chi_d)$ and $S_{63}(\chi_d)$ are also due to Lerch [44, p. 414]. In the terminology of class numbers, Holden [36] has established (6.4)-(6.6) in the associated special cases. Some results related to (6.1)-(6.3) were also found by Holden [39].

7. Sums over intervals of length $k/8$.

THEOREM 7.1. Let χ be even, let $\chi_{4k} = \chi_4 \chi$, and let $\chi_{8k} = \chi_4 \chi_8 \chi$. Then

$$(7.1)\quad S_{81} = \frac{G(\chi)}{2\pi} \{ \bar{\chi}(2) L(1, \bar{\chi}_{4k}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \},$$

$$S_{82} = \frac{G(\chi)}{2\pi} \{ [2 - \bar{\chi}(2)] L(1, \bar{\chi}_{4k}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \},$$

$$S_{83} = \frac{G(\chi)}{2\pi} \{ - [2 + \bar{\chi}(2)] L(1, \bar{\chi}_{4k}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi} \{ \bar{\chi}(2) L(1, \bar{\chi}_{4k}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \}.$$

Let χ be odd and let $\chi_{8k} = \chi_8 \chi$. Then

$$(7.2)\quad S_{81} = \frac{G(\chi)}{2\pi i} \left\{ \left[2 + \frac{1}{2} \bar{\chi}(4) \{ 1 - \bar{\chi}(2) \} \right] L(1, \bar{\chi}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{82} = \frac{G(\chi)}{2\pi i} \left\{ \bar{\chi}(2) \left[1 - \frac{3}{2} \bar{\chi}(2) + \frac{1}{2} \bar{\chi}(4) \right] L(1, \bar{\chi}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

$$S_{83} = \frac{G(\chi)}{2\pi i} \left\{ \bar{\chi}(2) \left[-1 + \frac{3}{2} \bar{\chi}(2) - \frac{1}{2} \bar{\chi}(4) \right] L(1, \bar{\chi}) + 2^{1/2} L(1, \bar{\chi}_{8k}) \right\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi i} \left\{ \left[2 - \frac{1}{2} \bar{\chi}(4) \right] [1 - \bar{\chi}(2)] L(1, \bar{\chi}) - 2^{1/2} L(1, \bar{\chi}_{8k}) \right\}.$$

We need only prove (7.1) and (7.2), for the remaining formulae can then be deduced from (7.1), (7.2), Theorem 3.2, Theorem 3.7, and elementary considerations. Since the proofs are similar to those in previous sections, we omit them. For the same reasons, proofs in sections 8-11 will not be given.