4. Sums over intervals of length $k/3$.

THEOREM 4.1. If χ is even and $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, then

$$S_{31} = \frac{3^{1/2} G(\chi)}{2\pi} L(1, \bar{\chi}_{3k}); \tag{4.1}$$

if χ is odd, then

$$S_{31} = \frac{G(\chi)}{2\pi i} \{3 - \bar{\chi}(3)\} L(1, \bar{\chi}). \tag{4.2}$$

Proof. First, suppose that χ is even. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant x < 2\pi/3, \\ 1/2, & x = 2\pi/3, \\ 0, & 2\pi/3 < x \leqslant \pi, \end{cases}$$

be an even function with period 2π. Then, by an elementary calculation,

$$f(x) = \frac{2}{3} + \frac{3^{1/2}}{\pi} \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{\cos(nx)}{n} \qquad (-\infty < x < \infty). \tag{4.3}$$

Now, multiply both sides of (2.1) by $3^{1/2} \left(\frac{n}{3}\right) / (\pi n)$ and sum on n, $1 \leq n < \infty$. With the use of (4.3), we obtain

$$\begin{aligned} 2S_{31} &= \sum_{j=1}^{k-1} \chi(j) \{ f(2\pi j/k) - 2/3 \} \\ &= \frac{3^{1/2}}{\pi} G(\chi) \sum_{n=1}^{\infty} \bar{\chi}(n) \left(\frac{n}{3}\right) \frac{1}{n} = \frac{3^{1/2}}{\pi} G(\chi) L(1, \bar{\chi}_{3k}), \end{aligned}$$

which completes the proof of (4.1).

For variety, we shall prove (4.2) by contour integration. Of course, the method of Fourier series used above works equally well here.

Let

$$f(z) = \frac{\pi F(z, \chi)}{z \sin \pi (z + 1/3)},$$

where

$$\begin{aligned} F(z, \chi) = 2i &\sum_{0 < j < k/3} \chi(j) \sin(\pi z + \pi/3 - 6\pi j z/k) \\ &+ e^{-3\pi i z} \sum_{k/3 < j < 2k/3} \chi(j) e^{6\pi i j z/k}. \end{aligned}$$

Observe that

$$R(f, 0) = \frac{\pi F(0, \chi)}{\sin(\pi/3)} = 2\pi i S_{31} \tag{4.4}$$

and that

$$R(f, n-1/3) = \frac{3(-1)^n}{3n-1} F(n-1/3, \chi) \tag{4.5}$$

$$= -\frac{3}{3n-1} G(3n-1, \chi) = -\frac{3}{3n-1} \bar{\chi}(3n-1) G(\chi),$$

by (2.1), where $-\infty < n < \infty$.

We integrate f over the same rectangle C_N as in the proof of Theorem 3.2. The estimate (3.8) is obtained by the same type of argument as in that proof. Applying the residue theorem, employing (4.4) and (4.5), and letting N tend to ∞, we deduce that

$$0 = 2\pi i\, S_{31} - 3G(\chi) \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1}$$

$$= 2\pi i\, S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1} + \sum_{n=0}^{\infty} \frac{\bar{\chi}(3n+1)}{3n+1} \right\}$$

$$= 2\pi i\, S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} - \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n)}{3n} \right\},$$

from which (4.2) readily follows.

Corollary 4.2. For any real primitive character χ with $k > 3$, $S_{31} > 0$.

Corollary 4.3. If $d > 0$ and $3 \nmid d$, then

$$S_{31}(\chi_d) = \frac{1}{2} h(-3d); \tag{4.6}$$

if $d < 0$, then

$$S_{31}(\chi_{-d}) = \frac{1}{2} \left\{ 3 - \left(\frac{d}{3}\right) \right\} h(d). \tag{4.7}$$

Corollary 4.4. Let $p > 3$. If $p \equiv 1 \pmod{12}$, then $h(-3p) \equiv 0 \pmod 4$, while if $p \equiv 5 \pmod{12}$, then $h(-3p) \equiv 2 \pmod 4$. If $p \equiv 3 \pmod 4$, then $h(-12p) \equiv 0 \pmod 4$. For any odd prime p, $h(-24p) \equiv 0 \pmod 4$.

Proof. Let $p = 6m + j$, where $j = 1$ or 5 and m is a non-negative integer. The number of summands in $S_{31}(\chi_p)$ is thus $2m + [j/3]$. The two congruences for $h(-3p)$ are then consequences of (4.6). The number of summands in $S_{31}(\chi_{4p})$ is $8m + [4j/3]$. If $p \equiv 7 \pmod{12}$, the number of non-zero summands is $4m$; if $p \equiv 11 \pmod{12}$, the number of non-zero summands is $4m + 2$. In either case, the number of non-zero summands is even, and so it follows from (4.6) that $h(-12p) \equiv 0 \pmod 4$ when $p \equiv 3 \pmod 4$. Lastly, the number of summands in $S_{31}(\chi_{8p})$ is $16m + [8j/3]$. If $j = 1$, there are $8m$ non-zero summands; if $j = 5$, there are $8m + 6$ non-zero summands. In either case, $S_{31}(\chi_{8p})$ is even, and we deduce from (4.6) that $h(-24p) \equiv 0 \pmod 4$.

Corollary 4.5. Let p and q be distinct primes with $p, q > 3$ and $p \equiv q \pmod 4$. Then $h(-3pq) \equiv 0 \pmod 4$.

Proof. Let $p = 6m + j$ and $q = 6m' + j'$, where $j, j' = 1$ or 5 and m and m' are non-negative integers. The number of summands in $S_{31}(\chi_{pq})$ is $[pq/3]$, and we observe that $[pq/3] \equiv [jj'/3] \pmod 2$. Of these summands, $[q/3] = 2m' + [j'/3]$ are multiples of p, and $[p/3] = 2m + [j/3]$ are multiples of q. Thus,

$$S_{31}(\chi_{pq}) \equiv [jj'/3] - [j'/3] - [j/3] \pmod 2 .$$

By examining all of the possibilities for the pair j, j', we find that $S_{31}(\chi_{pq})$ is always even. The result now follows from (4.6).

It is clear that the same type of argument yields congruences from $h(-12pq)$ and $h(-24pq)$.

The class number formulae (4.6) and (4.7) appear to be due originally to Lerch [44, pp. 402, 408]. Holden [36] has also given a proof of (4.7).

5. Sums over intervals of length $k/5$.

Theorem 5.1. Let χ be odd and let $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$. Then

$$(5.1) \qquad S_{51} = \frac{1}{4\pi i} G(\chi) \{ (5 - \bar{\chi}(5)) L(1, \bar{\chi}) - 5^{1/2} L(1, \bar{\chi}_{5k}) \}$$

and

$$(5.2) \qquad S_{52} = \frac{1}{2\pi i} 5^{1/2} G(\chi) L(1, \bar{\chi}_{5k}) .$$