Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CLASSICAL THEOREMS ON QUADRATIC RESIDUES
Autor: Berndt, Bruce C.
Kapitel: 4. SUMS OVER INTERVALS OF LENGTH k/3.
DOI: https://doi.org/10.5169/seals-48188

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

4. Sums over intervals of length $k/3$.

THEOREM 4.1. If χ is even and $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, then

$$S_{31} = \frac{3^{1/2} G(\chi)}{2\pi} L(1, \bar{\chi}_{3k}) ; \tag{4.1}$$

if χ is odd, then

$$S_{31} = \frac{G(\chi)}{2\pi i} \{ 3 - \bar{\chi}(3) \} L(1, \bar{\chi}) . \tag{4.2}$$

Proof. First, suppose that χ is even. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant x < 2\pi/3, \\ 1/2, & x = 2\pi/3, \\ 0, & 2\pi/3 < x \leqslant \pi, \end{cases}$$

be an even function with period 2π. Then, by an elementary calculation,

$$f(x) = \frac{2}{3} + \frac{3^{1/2}}{\pi} \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{\cos(nx)}{n} \qquad (-\infty < x < \infty) . \tag{4.3}$$

Now, multiply both sides of (2.1) by $3^{1/2} \left(\frac{n}{3}\right) / (\pi n)$ and sum on n, $1 \leqq n < \infty$. With the use of (4.3), we obtain

$$\begin{aligned} 2S_{31} &= \sum_{j=1}^{k-1} \chi(j) \{ f(2\pi j/k) - 2/3 \} \\ &= \frac{3^{1/2}}{\pi} G(\chi) \sum_{n=1}^{\infty} \bar{\chi}(n) \left(\frac{n}{3}\right) \frac{1}{n} = \frac{3^{1/2}}{\pi} G(\chi) L(1, \bar{\chi}_{3k}) , \end{aligned}$$

which completes the proof of (4.1).

For variety, we shall prove (4.2) by contour integration. Of course, the method of Fourier series used above works equally well here.

Let

$$f(z) = \frac{\pi F(z, \chi)}{z \sin \pi (z + 1/3)} ,$$

where

$$\begin{aligned} F(z, \chi) = 2i & \sum_{0 < j < k/3} \chi(j) \sin(\pi z + \pi/3 - 6\pi j z/k) \\ & + e^{-3\pi i z} \sum_{k/3 < j < 2k/3} \chi(j) e^{6\pi i j z/k} . \end{aligned}$$

Observe that

$$R(f, 0) = \frac{\pi F(0, \chi)}{\sin(\pi/3)} = 2\pi i S_{31} \tag{4.4}$$

and that

$$R(f, n-1/3) = \frac{3(-1)^n}{3n-1} F(n-1/3, \chi) \tag{4.5}$$

$$= -\frac{3}{3n-1} G(3n-1, \chi) = -\frac{3}{3n-1} \bar{\chi}(3n-1) G(\chi),$$

by (2.1), where $-\infty < n < \infty$.

We integrate f over the same rectangle C_N as in the proof of Theorem 3.2. The estimate (3.8) is obtained by the same type of argument as in that proof. Applying the residue theorem, employing (4.4) and (4.5), and letting N tend to ∞, we deduce that

$$0 = 2\pi i\, S_{31} - 3G(\chi) \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1}$$

$$= 2\pi i\, S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1} + \sum_{n=0}^{\infty} \frac{\bar{\chi}(3n+1)}{3n+1} \right\}$$

$$= 2\pi i\, S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} - \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n)}{3n} \right\},$$

from which (4.2) readily follows.

Corollary 4.2. For any real primitive character χ with $k > 3$, $S_{31} > 0$.

Corollary 4.3. If $d > 0$ and $3 \nmid d$, then

$$S_{31}(\chi_d) = \frac{1}{2} h(-3d); \tag{4.6}$$

if $d < 0$, then

$$S_{31}(\chi_{-d}) = \frac{1}{2} \left\{ 3 - \left(\frac{d}{3}\right) \right\} h(d). \tag{4.7}$$

Corollary 4.4. Let $p > 3$. If $p \equiv 1 \pmod{12}$, then $h(-3p) \equiv 0 \pmod 4$, while if $p \equiv 5 \pmod{12}$, then $h(-3p) \equiv 2 \pmod 4$. If $p \equiv 3 \pmod 4$, then $h(-12p) \equiv 0 \pmod 4$. For any odd prime p, $h(-24p) \equiv 0 \pmod 4$.

Proof. Let $p = 6m + j$, where $j = 1$ or 5 and m is a non-negative integer. The number of summands in $S_{31}(\chi_p)$ is thus $2m + [j/3]$. The two congruences for $h(-3p)$ are then consequences of (4.6). The number of summands in $S_{31}(\chi_{4p})$ is $8m + [4j/3]$. If $p \equiv 7 \pmod{12}$, the number of non-zero summands is $4m$; if $p \equiv 11 \pmod{12}$, the number of non-zero summands is $4m + 2$. In either case, the number of non-zero summands is even, and so it follows from (4.6) that $h(-12p) \equiv 0 \pmod 4$ when $p \equiv 3 \pmod 4$. Lastly, the number of summands in $S_{31}(\chi_{8p})$ is $16m + [8j/3]$. If $j = 1$, there are $8m$ non-zero summands; if $j = 5$, there are $8m + 6$ non-zero summands. In either case, $S_{31}(\chi_{8p})$ is even, and we deduce from (4.6) that $h(-24p) \equiv 0 \pmod 4$.

COROLLARY 4.5. Let p and q be distinct primes with $p, q > 3$ and $p \equiv q \pmod 4$. Then $h(-3pq) \equiv 0 \pmod 4$.

Proof. Let $p = 6m + j$ and $q = 6m' + j'$, where $j, j' = 1$ or 5 and m and m' are non-negative integers. The number of summands in $S_{31}(\chi_{pq})$ is $[pq/3]$, and we observe that $[pq/3] \equiv [jj'/3] \pmod 2$. Of these summands, $[q/3] = 2m' + [j'/3]$ are multiples of p, and $[p/3] = 2m + [j/3]$ are multiples of q. Thus,

$$S_{31}(\chi_{pq}) \equiv [jj'/3] - [j'/3] - [j/3] \pmod 2 .$$

By examining all of the possibilities for the pair j, j', we find that $S_{31}(\chi_{pq})$ is always even. The result now follows from (4.6).

It is clear that the same type of argument yields congruences from $h(-12pq)$ and $h(-24pq)$.

The class number formulae (4.6) and (4.7) appear to be due originally to Lerch [44, pp. 402, 408]. Holden [36] has also given a proof of (4.7).

5. SUMS OVER INTERVALS OF LENGTH $k/5$.

THEOREM 5.1. Let χ be odd and let $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$. Then

$$S_{51} = \frac{1}{4\pi i} G(\chi) \{ (5 - \bar{\chi}(5)) L(1, \bar{\chi}) - 5^{1/2} L(1, \bar{\chi}_{5k}) \} \tag{5.1}$$

and

$$S_{52} = \frac{1}{2\pi i} 5^{1/2} G(\chi) L(1, \bar{\chi}_{5k}) . \tag{5.2}$$