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4. SUMS OVER INTERVALS OF LENGTH  $k/3$ .

THEOREM 4.1. If  $\chi$  is even and  $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$ , then

$$(4.1) \quad S_{31} = \frac{3^{1/2} G(\chi)}{2\pi} L(1, \bar{\chi}_{3k});$$

if  $\chi$  is odd, then

$$(4.2) \quad S_{31} = \frac{G(\chi)}{2\pi i} \{3 - \bar{\chi}(3)\} L(1, \bar{\chi}).$$

*Proof.* First, suppose that  $\chi$  is even. Let

$$f(x) = \begin{cases} 1, & 0 \leq x < 2\pi/3, \\ 1/2, & x = 2\pi/3, \\ 0, & 2\pi/3 < x \leq \pi, \end{cases}$$

be an even function with period  $2\pi$ . Then, by an elementary calculation,

$$(4.3) \quad f(x) = \frac{2}{3} + \frac{3^{1/2}}{\pi} \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{\cos(nx)}{n} \quad (-\infty < x < \infty).$$

Now, multiply both sides of (2.1) by  $3^{1/2} \left(\frac{n}{3}\right) / (\pi n)$  and sum on  $n$ ,  $1 \leq n < \infty$ .

With the use of (4.3), we obtain

$$\begin{aligned} 2S_{31} &= \sum_{j=1}^{k-1} \chi(j) \{f(2\pi j/k) - 2/3\} \\ &= \frac{3^{1/2}}{\pi} G(\chi) \sum_{n=1}^{\infty} \bar{\chi}(n) \left(\frac{n}{3}\right) \frac{1}{n} = \frac{3^{1/2}}{\pi} G(\chi) L(1, \bar{\chi}_{3k}), \end{aligned}$$

which completes the proof of (4.1).

For variety, we shall prove (4.2) by contour integration. Of course, the method of Fourier series used above works equally well here.

Let

$$f(z) = \frac{\pi F(z, \chi)}{z \sin \pi(z + 1/3)},$$

where

$$\begin{aligned} F(z, \chi) &= 2i \sum_{0 < j < k/3} \chi(j) \sin(\pi z + \pi/3 - 6\pi jz/k) \\ &\quad + e^{-3\pi iz} \sum_{k/3 < j < 2k/3} \chi(j) e^{6\pi ijz/k}. \end{aligned}$$

Observe that

$$(4.4) \quad R(f, 0) = \frac{\pi F(0, \chi)}{\sin(\pi/3)} = 2\pi i S_{31}$$

and that

$$(4.5) \quad \begin{aligned} R(f, n-1/3) &= \frac{3(-1)^n}{3n-1} F(n-1/3, \chi) \\ &= -\frac{3}{3n-1} G(3n-1, \chi) = -\frac{3}{3n-1} \bar{\chi}(3n-1) G(\chi), \end{aligned}$$

by (2.1), where  $-\infty < n < \infty$ .

We integrate  $f$  over the same rectangle  $C_N$  as in the proof of Theorem 3.2. The estimate (3.8) is obtained by the same type of argument as in that proof. Applying the residue theorem, employing (4.4) and (4.5), and letting  $N$  tend to  $\infty$ , we deduce that

$$\begin{aligned} 0 &= 2\pi i S_{31} - 3G(\chi) \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1} \\ &= 2\pi i S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1} + \sum_{n=0}^{\infty} \frac{\bar{\chi}(3n+1)}{3n+1} \right\} \\ &= 2\pi i S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} - \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n)}{3n} \right\}, \end{aligned}$$

from which (4.2) readily follows.

**COROLLARY 4.2.** For any real primitive character  $\chi$  with  $k > 3$ ,  $S_{31} > 0$ .

**COROLLARY 4.3.** If  $d > 0$  and  $3 \nmid d$ , then

$$(4.6) \quad S_{31}(\chi_d) = \frac{1}{2} h(-3d);$$

if  $d < 0$ , then

$$(4.7) \quad S_{31}(\chi_{-d}) = \frac{1}{2} \left\{ 3 - \left(\frac{d}{3}\right) \right\} h(d).$$

**COROLLARY 4.4.** Let  $p > 3$ . If  $p \equiv 1 \pmod{12}$ , then  $h(-3p) \equiv 0 \pmod{4}$ , while if  $p \equiv 5 \pmod{12}$ , then  $h(-3p) \equiv 2 \pmod{4}$ . If  $p \equiv 3 \pmod{4}$ , then  $h(-12p) \equiv 0 \pmod{4}$ . For any odd prime  $p$ ,  $h(-24p) \equiv 0 \pmod{4}$ .

*Proof.* Let  $p = 6m + j$ , where  $j = 1$  or  $5$  and  $m$  is a non-negative integer. The number of summands in  $S_{31}(\chi_p)$  is thus  $2m + [j/3]$ . The two congru-

ences for  $h(-3p)$  are then consequences of (4.6). The number of summands in  $S_{31}(\chi_{4p})$  is  $8m + [4j/3]$ . If  $p \equiv 7 \pmod{12}$ , the number of non-zero summands is  $4m$ ; if  $p \equiv 11 \pmod{12}$ , the number of non-zero summands is  $4m + 2$ . In either case, the number of non-zero summands is even, and so it follows from (4.6) that  $h(-12p) \equiv 0 \pmod{4}$  when  $p \equiv 3 \pmod{4}$ . Lastly, the number of summands in  $S_{31}(\chi_{8p})$  is  $16m + [8j/3]$ . If  $j = 1$ , there are  $8m$  non-zero summands; if  $j = 5$ , there are  $8m + 6$  non-zero summands. In either case,  $S_{31}(\chi_{8p})$  is even, and we deduce from (4.6) that  $h(-24p) \equiv 0 \pmod{4}$ .

**COROLLARY 4.5.** Let  $p$  and  $q$  be distinct primes with  $p, q > 3$  and  $p \equiv q \pmod{4}$ . Then  $h(-3pq) \equiv 0 \pmod{4}$ .

*Proof.* Let  $p = 6m + j$  and  $q = 6m' + j'$ , where  $j, j' = 1$  or  $5$  and  $m$  and  $m'$  are non-negative integers. The number of summands in  $S_{31}(\chi_{pq})$  is  $[pq/3]$ , and we observe that  $[pq/3] \equiv [jj'/3] \pmod{2}$ . Of these summands,  $[q/3] = 2m' + [j'/3]$  are multiples of  $p$ , and  $[p/3] = 2m + [j/3]$  are multiples of  $q$ . Thus,

$$S_{31}(\chi_{pq}) \equiv [jj'/3] - [j'/3] - [j/3] \pmod{2}.$$

By examining all of the possibilities for the pair  $j, j'$ , we find that  $S_{31}(\chi_{pq})$  is always even. The result now follows from (4.6).

It is clear that the same type of argument yields congruences from  $h(-12pq)$  and  $h(-24pq)$ .

The class number formulae (4.6) and (4.7) appear to be due originally to Lerch [44, pp. 402, 408]. Holden [36] has also given a proof of (4.7).

## 5. SUMS OVER INTERVALS OF LENGTH $k/5$ .

**THEOREM 5.1.** Let  $\chi$  be odd and let  $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$ . Then

$$(5.1) \quad S_{51} = \frac{1}{4\pi i} G(\chi) \{ (5 - \bar{\chi}(5)) L(1, \bar{\chi}) - 5^{1/2} L(1, \bar{\chi}_{5k}) \}$$

and

$$(5.2) \quad S_{52} = \frac{1}{2\pi i} 5^{1/2} G(\chi) L(1, \bar{\chi}_{5k}).$$