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prove (1.1). The fourth method is similar to the third and uses character analogues of the Poisson summation formula which have been established in various versions by Berger [5], Lerch [44], Mordell [46], Guinand [30], the author [6], and Schoenfeld and the author [9]. The application of the character Poisson formula to problems of this type appears to be new. However, Yamamoto [62] has recently used essentially the same technique to derive some of the results of this paper. The method is also briefly described by the author in [7].

In most cases, we have chosen a direct, analytic method of proof, whereas a possibly less direct but more elementary argument *with* the use of Dirichlet's main theorems is possible. In fact, throughout the literature, the latter attack is generally the tact that is chosen. In particular, see the aforementioned papers of Holden and Karpinski and a paper of Rédei [57].

The author is very grateful to his colleague Samuel Wagstaff, Jr. who computed lengthy tables of sums of the Legendre symbol. These computations were immensely helpful to the author in formulating conjectures and testing conjectures. The author is also very grateful to Duncan Buell for extensive calculations in connection with some inequalities for class numbers conjectured by the author. (See section 14.)

2. Notation and preliminary results

Throughout the sequel, χ shall denote a non-principal, primitive character of modulus k. To indicate the dependence upon the modulus k, we shall often write χ_k for χ. Always, p denotes an odd prime. If $p_1, \ldots, p_r$ denote distinct odd primes, let

$$d = \pm 2^{\alpha} \prod_{i=1}^{r} (-1)^{(p_i-1)/2} p_i .$$

Here, $r \geqq 0$ and $\alpha = 0$, 2 or 3; if $\alpha = 0$, then $r > 0$ and the plus sign must be taken, if $\alpha = 2$, the minus sign must be taken, and if $\alpha = 3$, either sign may be taken. If n is a positive integer, let $\left(\frac{d}{n}\right)$ denote the Kronecker symbol. Every real primitive character is of the form $\left(\frac{d}{n}\right)$, and the modulus of each such character is $|d|$ [20, p. 42]. Furthermore, $\left(\frac{d}{n}\right)$ is even or odd according to whether $d > 0$ or $d < 0$, respectively.

The following real primitive characters shall frequently arise in the sequel. Let

$$\chi_4(n) = \begin{cases} (-1)^{(n-1)/2}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases}$$

$$\chi_8(n) = \begin{cases} (-1)^{(n^2-1)/8}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases}$$

and $\chi_4\chi_8(n) = \chi_4(n)\chi_8(n)$. We shall often write, for example, $\chi_{4k}(n) = \chi_k(n)\chi_4(n)$. However, possibly the modulus of $\chi_k(n)\chi_4(n)$ is *not* $4k$. It will be understood, nonetheless, that despite the notation χ_{4k}, the least period shall be taken to be the modulus of $\chi_k(n)\chi_4(n)$.

Let $G(n, \chi)$ denote the Gauss sum

$$G(n, \chi) = \sum_{j \bmod k} \chi(j) e^{2\pi i n j/k},$$

and put $G(\chi) = G(1, \chi)$. We shall need the fundamental property [2, p. 312]

$$G(n, \chi) = \bar{\chi}(n) G(\chi). \tag{2.1}$$

Furthermore, if $\chi(n) = \left(\frac{d}{n}\right)$, we have [2, p. 319]

$$G(\chi) = \begin{cases} d^{1/2}, & \text{if } d > 0, \\ i\,|d|^{1/2}, & \text{if } d < 0. \end{cases} \tag{2.2}$$

As usual, $L(s, \chi)$ denotes the Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \qquad (\operatorname{Re} s > 0). \tag{2.3}$$

The connection between L-functions and class numbers of imaginary quadratic fields is given by the basic formula [2, p. 295], [31, p. 395].

$$h(d) = \frac{|d|^{1/2}}{\pi} L(1, \chi_{-d}), \tag{2.4}$$

where $d \leqq -7$, which we shall always assume in the sequel.

The sums that we shall consider are

$$S_{ji} = S_{ji}(\chi) = \sum_{(i-1)k/j < n < ik/j} \chi(n),$$

where i and j are natural numbers, and k is the modulus of χ.

Lastly, the residue of a meromorphic function f at a pole z_0 shall always be denoted by $R(f, z_0)$.