Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 22 (1976)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: CLASSICAL THEOREMS ON QUADRATIC RESIDUES
Autor: Berndt, Bruce C.
Kapitel: 2. Notation and preliminary results
DOI: https://doi.org/10.5169/seals-48188

Nutzungsbedingungen
Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation
L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use
The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 15.02.2026

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

prove (1.1). The fourth method is similar to the third and uses character analogues of the Poisson summation formula which have been established in various versions by Berger [5], Lerch [44], Mordell [46], Guinand [30], the author [6], and Schoenfeld and the author [9]. The application of the character Poisson formula to problems of this type appears to be new. However, Yamamoto [62] has recently used essentially the same technique to derive some of the results of this paper. The method is also briefly described by the author in [7].

In most cases, we have chosen a direct, analytic method of proof, whereas a possibly less direct but more elementary argument *with* the use of Dirichlet's main theorems is possible. In fact, throughout the literature, the latter attack is generally the tact that is chosen. In particular, see the aforementioned papers of Holden and Karpinski and a paper of Rédei [57].

The author is very grateful to his colleague Samuel Wagstaff, Jr. who computed lengthy tables of sums of the Legendre symbol. These computations were immensely helpful to the author in formulating conjectures and testing conjectures. The author is also very grateful to Duncan Buell for extensive calculations in connection with some inequalities for class numbers conjectured by the author. (See section 14.)

2. Notation and preliminary results

Throughout the sequel, χ shall denote a non-principal, primitive character of modulus k. To indicate the dependence upon the modulus k, we shall often write χ_k for χ. Always, p denotes an odd prime. If $p_1, \ldots, p_r$ denote distinct odd primes, let

$$d = \pm 2^\alpha \prod_{i=1}^{r} (-1)^{(p_i-1)/2} p_i .$$

Here, $r \geqq 0$ and $\alpha = 0$, 2 or 3; if $\alpha = 0$, then $r > 0$ and the plus sign must be taken, if $\alpha = 2$, the minus sign must be taken, and if $\alpha = 3$, either sign may be taken. If n is a positive integer, let $\left(\frac{d}{n}\right)$ denote the Kronecker symbol. Every real primitive character is of the form $\left(\frac{d}{n}\right)$, and the modulus of each such character is $|d|$ [20, p. 42]. Furthermore, $\left(\frac{d}{n}\right)$ is even or odd according to whether $d > 0$ or $d < 0$, respectively.

The following real primitive characters shall frequently arise in the sequel. Let

$$\chi_4(n) = \begin{cases} (-1)^{(n-1)/2}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases}$$

$$\chi_8(n) = \begin{cases} (-1)^{(n^2-1)/8}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases}$$

and $\chi_4\chi_8(n) = \chi_4(n)\chi_8(n)$. We shall often write, for example, $\chi_{4k}(n) = \chi_k(n)\chi_4(n)$. However, possibly the modulus of $\chi_k(n)\chi_4(n)$ is *not* $4k$. It will be understood, nonetheless, that despite the notation χ_{4k}, the least period shall be taken to be the modulus of $\chi_k(n)\chi_4(n)$.

Let $G(n, \chi)$ denote the Gauss sum

$$G(n, \chi) = \sum_{j \bmod k} \chi(j) e^{2\pi i n j/k},$$

and put $G(\chi) = G(1, \chi)$. We shall need the fundamental property [2, p. 312]

$$G(n, \chi) = \bar{\chi}(n) G(\chi). \tag{2.1}$$

Furthermore, if $\chi(n) = \left(\frac{d}{n}\right)$, we have [2, p. 319]

$$G(\chi) = \begin{cases} d^{1/2}, & \text{if } d > 0, \\ i\,|d|^{1/2}, & \text{if } d < 0. \end{cases} \tag{2.2}$$

As usual, $L(s, \chi)$ denotes the Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \quad (\mathrm{Re}\ s > 0). \tag{2.3}$$

The connection between L-functions and class numbers of imaginary quadratic fields is given by the basic formula [2, p. 295], [31, p. 395].

$$h(d) = \frac{|d|^{1/2}}{\pi} L(1, \chi_{-d}), \tag{2.4}$$

where $d \leq -7$, which we shall always assume in the sequel.

The sums that we shall consider are

$$S_{ji} = S_{ji}(\chi) = \sum_{(i-1)k/j < n < ik/j} \chi(n),$$

where i and j are natural numbers, and k is the modulus of χ.

Lastly, the residue of a meromorphic function f at a pole z_0 shall always be denoted by $R(f, z_0)$.