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CLASSICAL THEOREMS ON QUADRATIC RESIDUES

by Bruce C. BERNDT

1. INTRODUCTION

In 1839, Dirichlet [23] proved that if p is a prime with $p \equiv 3 \pmod 4$, then

(1.1) $$\sum_{0<n<p/2} \left(\frac{n}{p}\right) > 0\,,$$

where $\left(\frac{n}{p}\right)$ denotes the Legendre symbol. In other words, the number of quadratic residues in the interval $(0, p/2)$ always exceeds the number of quadratic non-residues in that interval. Dirichlet's deduction of (1.1) was an immediate consequence of one of his class number formulas for binary quadratic forms. All known proofs of (1.1) are nonelementary in that they use infinite series. Many authors have expressed the desire for a truely elementary proof of (1.1). In fact, Landau [43, p. 129] remarks "Aber noch kein Mensch hat diese wahre Tatsache mit elementaren Mitteln beweisen können." Although we give some new proofs of (1.1) here, unfortunately, none can be considered elementary.

Another result with its origins in a class number formula of Dirichlet is the following. If p is a prime with $p \equiv 1 \pmod 4$, then

(1.2) $$\sum_{0<n<p/4} \left(\frac{n}{p}\right) > 0\,.$$

Thus, the number of quadratic residues in the interval $(0, p/4)$ always exceeds the number of quadratic non-residues there. As with (1.1), an elementary proof of (1.2) does not exist. Furthermore, (1.2) does not appear to be as widely known as (1.1). All published proofs of (1.2) follow from class number formulas. We give here some proofs of (1.2) that do not involve class number considerations, although, admittedly, the use of L-functions gives an undeniable link with class numbers.

The main purpose of this study is to make a systematically thorough attempt to discover which sums of the Legendre symbol, or more generally,

sums of real primitive characters, are always positive (or negative). In other words, on which intervals for which classes of primes are results like (1.1) and (1.2) possible ? The quadratic excess on (a, b) is defined to be $\sum_{a<n<b} \left(\frac{n}{p}\right)$
Thus, for example, if $p > 3$ is prime, we show that the quadratic excess on $(0, p/3)$ is always positive. If $p \equiv 11, 19 \pmod{40}$, then the quadratic excess on $(0, p/10)$ is positive. If $p \equiv 5 \pmod{24}$, then the quadratic excess on $(3p/8, 5p/12)$ is negative. We establish many results of this type. Many of our results are not new and can be found scattered throughout the literature since 1839. In particular, Lerch [44], Holden [36-39], and Karpinski [42] have established many of the results proved here. However, a goodly number of our findings appear to be new. Moreover, our results are most often proven with greater generality than elsewhere in the literature.

Many intervals are found for which the quadratic excess is zero. Such results, however, can invariably be proved by purely elementary techniques. Many examples of this sort of result may be found in a paper by Chowla and the author [8] and, even moreso, in the work of Johnson and Mitchell [41]. A related question is examined in a paper of Wolke [61].

Let $h(d)$ denote the class number of the quadratic field of discriminant d over the rational numbers. For $d < 0$, we obtain many congruences for class numbers as easy corollaries of our efforts to find positive character sums. Again, many of these results are scattered throughout the literature, but many do not appear to have been previously noticed. As an example of the type of result obtained, we state a lemma of Stark [59] which was important in his proof that there are exactly 9 imaginary quadratic fields of class number 1. If p is a prime with $p \equiv 19 \pmod{24}$, then $h(-12p) \equiv 4 \pmod{8}$. As other examples, we mention that if $p \equiv 7 \pmod{20}$, then $h(-5p) \equiv 2h(-p) \pmod{8}$; if $p \equiv 7 \pmod{24}$, then $h(-24p) \equiv 4 \pmod{8}$; and if $p \equiv 17 \pmod{48}$, then $h(-24p) - 2h(-8p) + 2h(-3p) \equiv 0 \pmod{16}$.

Our work involving congruences for class numbers overlaps considerably with that of Pizer [53]. However, the techniques are entirely dissimilar. Pizer uses the theory of type numbers of Eichler orders [52], while we use the theory of Dirichlet L-functions. Pizer [53] proves congruences for class numbers with discriminants containing three or fewer primes. We concentrate primarily on discriminants with just one odd prime or small multiples of one odd prime. It should be mentioned, however, that our methods are applicable to imaginary quadratic fields with discriminants containing any number of distinct odd prime factors. Perhaps Hurwitz [40] was the first to prove congruences for class numbers with discriminants involving two distinct prime factors. Brown [11], [12], [14] and Hasse [34], [35] have achieved several results for two distinct prime factors. For congruences relating class numbers for imaginary quadratic fields with discriminants containing three distinct prime factors, see,in particular, papers of Pumplün [55], Brown [11], and Brown and Parry [15]. Finally, the divisibility by a power of 2 of class numbers for imaginary quadratic fields with discriminants containing an arbitrary number of distinct odd primes has been studied by Plancherel [54], Rédei [56], and Rédei and Reichardt [58]. A related paper is [1].

An elementary argument [60] shows that (1.1) is equivalent to another theorem of Dirichlet [23]. Let p be a prime with $p \equiv 3 \pmod 4$. Let r denote an arbitrary quadratic residue and n an arbitrary quadratic non-residue modulo p in the interval $(0, p)$. Then

$$\sum_{0<n<p} n - \sum_{0<r<p} r > 0 . \tag{1.3}$$

In other words, the sum of the non-residues in $(0, p)$ always outweighs the sum of the residues in the same interval. In the penultimate section of this paper, many other results of this type are established. Most of these theorems appear to be new.

In the last section of the paper, we state several open problems and conjectures on positive sums of the Legendre symbol and on class numbers.

The organization for the paper is now briefly described. We shall, in turn, examine various intervals for which positivity results can be obtained. Our techniques are generally applicable to arbitrary primitive characters. Thus, for each class of intervals we first give theorems for arbitrary primitive characters that express character sums over these intervals in terms of L-functions. Next, we determine for real primitive characters when the character sum is always positive, negative, or zero. Thirdly, we translate our representations of real primitive character sums into statements involving class numbers. Fourthly, we deduce congruences for class numbers.

Our techniques can be classified into four main types. In section 3, we use the partial fraction decomposition of the cotangent function to effect a very simple proof of Dirichlet's theorem in the form (1.3). Our second technique uses contour integration and also appears to be completely new. The third technique uses Fourier series and is an extension of the method used, for example, by Dirichlet [24], Chowla [18], and Moser [47] to prove (1.1). The fourth method is similar to the third and uses character analogues of the Poisson summation formula which have been established in various versions by Berger [5], Lerch [44], Mordell [46], Guinand [30], the author [6], and Schoenfeld and the author [9]. The application of the character Poisson formula to problems of this type appears to be new. However, Yamamoto [62] has recently used essentially the same technique to derive some of the results of this paper. The method is also briefly described by the author in [7].

In most cases, we have chosen a direct, analytic method of proof, whereas a possibly less direct but more elementary argument *with* the use of Dirichlet's main theorems is possible. In fact, throughout the literature, the latter attack is generally the tact that is chosen. In particular, see the aforementioned papers of Holden and Karpinski and a paper of Rédei [57].

The author is very grateful to his colleague Samuel Wagstaff, Jr. who computed lengthy tables of sums of the Legendre symbol. These computations were immensely helpful to the author in formulating conjectures and testing conjectures. The author is also very grateful to Duncan Buell for extensive calculations in connection with some inequalities for class numbers conjectured by the author. (See section 14.)

2. Notation and preliminary results

Throughout the sequel, χ shall denote a non-principal, primitive character of modulus k. To indicate the dependence upon the modulus k, we shall often write χ_k for χ. Always, p denotes an odd prime. If $p_1, \ldots, p_r$ denote distinct odd primes, let

$$d = \pm 2^{\alpha} \prod_{i=1}^{r} (-1)^{(p_i-1)/2} p_i .$$

Here, $r \geq 0$ and $\alpha = 0$, 2 or 3; if $\alpha = 0$, then $r > 0$ and the plus sign must be taken, if $\alpha = 2$, the minus sign must be taken, and if $\alpha = 3$, either sign may be taken. If n is a positive integer, let $\left(\frac{d}{n}\right)$ denote the Kronecker symbol. Every real primitive character is of the form $\left(\frac{d}{n}\right)$, and the modulus of each such character is $|d|$ [20, p. 42]. Furthermore, $\left(\frac{d}{n}\right)$ is even or odd according to whether $d > 0$ or $d < 0$, respectively.

The following real primitive characters shall frequently arise in the sequel. Let

$$\chi_4(n) = \begin{cases} (-1)^{(n-1)/2}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases}$$

$$\chi_8(n) = \begin{cases} (-1)^{(n^2-1)/8}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases}$$

and $\chi_4\chi_8(n) = \chi_4(n)\chi_8(n)$. We shall often write, for example, $\chi_{4k}(n) = \chi_k(n)\chi_4(n)$. However, possibly the modulus of $\chi_k(n)\chi_4(n)$ is *not* $4k$. It will be understood, nonetheless, that despite the notation χ_{4k}, the least period shall be taken to be the modulus of $\chi_k(n)\chi_4(n)$.

Let $G(n, \chi)$ denote the Gauss sum

$$G(n, \chi) = \sum_{j \bmod k} \chi(j) e^{2\pi i n j/k},$$

and put $G(\chi) = G(1, \chi)$. We shall need the fundamental property [2, p. 312]

$$G(n, \chi) = \bar{\chi}(n) G(\chi). \tag{2.1}$$

Furthermore, if $\chi(n) = \left(\frac{d}{n}\right)$, we have [2, p. 319]

$$G(\chi) = \begin{cases} d^{1/2}, & \text{if } d > 0, \\ i\,|d|^{1/2}, & \text{if } d < 0. \end{cases} \tag{2.2}$$

As usual, $L(s, \chi)$ denotes the Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \qquad (\mathrm{Re}\ s > 0). \tag{2.3}$$

The connection between L-functions and class numbers of imaginary quadratic fields is given by the basic formula [2, p. 295], [31, p. 395].

$$h(d) = \frac{|d|^{1/2}}{\pi} L(1, \chi_{-d}), \tag{2.4}$$

where $d \leqq -7$, which we shall always assume in the sequel.

The sums that we shall consider are

$$S_{ji} = S_{ji}(\chi) = \sum_{(i-1)k/j < n < ik/j} \chi(n),$$

where i and j are natural numbers, and k is the modulus of χ.

Lastly, the residue of a meromorphic function f at a pole z_0 shall always be denoted by $R(f, z_0)$.

3. Dirichlet's fundamental theorems

Theorem 3.1. If p is a prime with $p \equiv 3 \pmod 4$, then (1.3) holds.

Proof. Let M denote the left side of (1.3). We shall first show that

$$M = \frac{1}{2} p^{1/2} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot (\pi k/p) . \tag{3.1}$$

Formula (3.1) is quite ancient, and several references to it can be found in Dickson's history [22, Chapter 6]. For references to more recent proofs and generalizations, see [7, section 5]. For completeness, we shall reproduce the following argument of Whiteman [60]. Since

$$\sum_{j=1}^{p-1} j \sin (2\pi jk/p) = -\frac{1}{2} p \cot (\pi k/p) , \tag{3.2}$$

we have, upon the use of (3.2) and then (2.2),

$$\begin{aligned}\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot (\pi k/p) &= -\frac{2}{p} \sum_{j=1}^{p-1} j \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \sin (2\pi jk/p) \\ &= -\frac{2}{p} \sum_{j=1}^{p-1} j \left(\frac{j}{p}\right) p^{1/2} ,\end{aligned}$$

and (3.1) immediately follows.

Thus, to show that M is positive, it suffices to show that the right side of (3.1) is positive. As

$$M = \sum_{j=1}^{p-1} j - 2 \sum_{1 \leqq r \leqq p-1} r \equiv p(p-1)/2 \equiv 1 \pmod 2 ,$$

since $p \equiv 3 \pmod 4$, it suffices to show that the right side of (3.1) is non-negative.

Using the partial fraction decomposition

$$\pi \cot (\pi x) = \lim_{N -> \infty} \sum_{m=-N}^{N} 1/(m+x) ,$$

where x is non-integral, we have

$$\begin{aligned}\sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \cot (\pi k/p) &= \frac{1}{\pi} \lim_{N -> \infty} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \sum_{m=-N}^{N} \frac{1}{m + k/p} \\ &= \frac{p}{\pi} \lim_{N -> \infty} \sum_{j=-Np}^{(N+1)p} \left(\frac{j}{p}\right) \frac{1}{j} \\ &= \frac{2p}{\pi} L(1, \chi_p) ,\end{aligned} \tag{3.3}$$

where in the penultimate step we put $j = mp + k$ and lastly use the fact that $\left(\frac{j}{p}\right)$ is an odd function of j. Thus, from (3.3), it suffices to show that $L(1, \chi_p)$ is non-negative.

Now, for $s > 1$,

$$L(s, \chi_p) = \prod_q \left\{1 - \left(\frac{q}{p}\right) q^{-s}\right\}^{-1}, \tag{3.4}$$

where the product is over all primes q. Each factor on the right side of (3.4) is positive for $s > 1$. Thus, $L(s, \chi_p) > 0$ for $s > 1$. Since the infinite series in (2.3) converges uniformly for $\varepsilon \leqq s < \infty$, where $0 < \varepsilon < 1$, $L(s, \chi_p)$ is continuous at $s = 1$. Hence, $L(1, \chi_p) \geqq 0$, and the proof of Theorem 3.1 is complete.

Apparently, Chung [19] was the first person to give a proof of Theorem 3.1 that was independent of the consideration of binary quadratic forms and class numbers. Subsequent proofs of (1.1) and (1.3) were given by Chowla [18], Whiteman [60], Moser [47] and Carlitz [16]. Moser also discusses (1.1) in [48]. There is also a nice proof of (1.3) in Davenport's book [20, p. 10]. All of these proofs use Fourier series. Now, in fact, the proofs of Chung, Chowla, Whiteman, Moser, and Carlitz are essentially no different from the proofs given by Dirichlet [24] in 1840 and later by Berger [5] in 1884 and Lerch [44] in 1905. The only difference is that the five aforementioned authors avoid the language of class numbers.

Perhaps our proof above is a modicum more elementary in that it does not use Fourier series but instead employs the partial fraction decomposition of cot (πx), which can be derived by quite elementary means [49]. Of course, our method above is applicable to any odd real primitive character.

Next, we show that very short proofs of (1.1) and (1.3) may be given by the use of contour integration.

THEOREM 3.2. If χ is odd, then

$$S_{21} = \frac{iG(\chi)}{\pi} \{\bar{\chi}(2) - 2\} L(1, \bar{\chi}).$$

Proof. Let

$$f(z) = \frac{\pi F(z, \chi)}{z \cos(\pi z)},$$

where

$$F(z, \chi) = \sum_{0<j<k/2} \chi(j) \cos(\pi z - 4\pi j z/k).$$

Observe that f has a simple pole at $z = 0$ with

$$(3.5) \qquad R(f, 0) = \pi F(0, \chi) = \pi S_{21} .$$

Also, f has simple poles at $z = (2n-1)/2, -\infty < n < \infty$, with

$$(3.6) \qquad \begin{aligned} R(f, (2n-1)/2) &= \frac{2(-1)^n}{2n-1} F((2n-1)/2, \chi) \\ &= \frac{i}{2n-1} G(2n-1, \chi) \\ &= \frac{i}{2n-1} \bar{\chi}(2n-1) G(\chi), \end{aligned}$$

by (2.1).

Let C_N denote the positively oriented rectangle with center at the origin, horizontal sides of length $2N$, and vertical sides of length $N^{1/2}$, where N is a positive integer. Applying the residue theorem with the aid of (3.5) and (3.6), we get

$$(3.7) \quad I_N \equiv \frac{1}{2\pi i} \int_{C_N} f(z)\, dz = \pi S_{21} + iG(\chi) \sum_{n=-N+1}^{N} \frac{\bar{\chi}(2n-1)}{2n-1} .$$

From the definition of $F(z, \chi)$, we see that there exists a positive constant A, independent of N, such that for all $z = x + iy$ on the horizontal sides of C_N, $| F(z, \chi)/\cos(\pi z) | \leqq A \exp(-2\pi |y|/k)$. Also, $F(z, \chi)/\cos(\pi z)$ has period $2k$. Thus, there is a positive constant B, independent of N, such that for all z on the vertical sides of C_N, $| F(z, \chi)/\cos(\pi z) | \leqq B$. Hence we find that as N tends to ∞,

$$(3.8) \qquad I_N = 0(e^{-\pi N^{1/2}/k}) + 0(N^{-1/2}) = o(1) .$$

Letting N tend to ∞, we deduce from (3.7) and (3.8) that

$$S_{21} = -\frac{iG(\chi)}{\pi} \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(2n-1)}{2n-1} = -\frac{2iG(\chi)}{\pi} \left\{1 - \frac{1}{2} \bar{\chi}(2)\right\} L(1, \bar{\chi}),$$

which completes the proof.

A direct proof of Theorem 3.1, or, more properly, an obvious generalization thereof, may also be achieved by contour integration. Integrate

$$\frac{1}{z(e^{2\pi i z} - 1)} \sum_{0 < j < p} \chi(j)\, e^{2\pi i j z/p}$$

over a rectangle C_N like that of the previous proof, but with the horizontal sides of length $2N + 1$.

A short proof of Theorem 3.2 using the character Poisson formula can be found in [7, section 4].

From the classical theory of L-functions, it can be shown that if χ is a real primitive character, then $L(1, \chi) > 0$ [2, pp. 27-28]. We shall repeatedly use this fact without comment in the sequel. Hence, the following is immediate from Theorem 3.2.

COROLLARY 3.3. If χ is real and odd, then $S_{21} > 0$.

The following corollary is an immediate consequence of Theorem 3.2 and (2.4) and is one of Dirichlet's famous class number formulas [23].

COROLLARY 3.4. If $d < 0$, then

$$S_{21} = \left\{2 - \left(\frac{d}{2}\right)\right\} h(d).$$

COROLLARY 3.5. If $p \equiv 3 \pmod 4$, then $S_{21}(\chi_p)$ is odd; if, furthermore, $p \equiv 3 \pmod 8$, then $3 \mid S_{21}(\chi_p)$.

COROLLARY 3.6. If $p \equiv 3 \pmod 4$, then $h(-p)$ is odd.

We now will give two proofs of (1.2) below. The first, in essence, is due to Dirichlet [24].

THEOREM 3.7. Let χ be even. Then if $\chi_{4k}(n) = \chi_4(n)\chi_k(n)$,

$$S_{41} = \frac{G(\chi)}{\pi} L(1, \bar{\chi}_{4k}).$$

First proof. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant x < \pi/2, \\ 0, & x = \pi/2, \\ -1, & \pi/2 < x \leqslant \pi, \end{cases}$$

be an even function with period 2π. Calculating the Fourier series of f, we find that

$$f(x) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(2n-1)x}{2n-1} \qquad (-\infty < x < \infty). \tag{3.9}$$

Next, in (2.1), replace n by $2n-1$. Then multiply both sides by $(-1)^n/(2n-1)$ and sum on n, $1 \leq n < \infty$, to get

$$(3.10)\qquad -G(\chi)\,L(1,\bar{\chi}_{4k}) = \sum_{j=1}^{k-1} \chi(j) \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \cos\{2\pi j(2n-1)/k\}$$

$$= -\frac{\pi}{4} \sum_{j=1}^{k-1} \chi(j) f(2\pi j/k),$$

by (3.9). Since χ is even, $S_{41} = -S_{42} = -S_{43} = S_{44}$. Using the definition of f, we see then that (3.10) reduces to

$$G(\chi)\,L(1,\bar{\chi}_{4k}) = \pi S_{41},$$

which completes the proof.

Second proof. Let

$$f(z) = \frac{\pi F(z,\chi)}{z \cos(\pi z)},$$

where

$$F(z,\chi) = \sum_{0<j<k/4} \chi(j) \cos(4\pi j z/k)$$

$$- \sum_{k/4<j<k/2} \chi(j) \cos(2\pi z - 4\pi j z/k).$$

Note that

$$(3.11)\qquad R(f,0) = \pi F(0,\chi) = \pi(S_{41} - S_{42}) = 2\pi S_{41}$$

and that, for $-\infty < n < \infty$,

$$(3.12)\qquad R(f,(2n-1)/2) = \frac{2(-1)^n}{2n-1} F((2n-1)/2,\chi)$$

$$= \frac{(-1)^n}{2n-1} G((2n-1)/2,\chi)$$

$$= \frac{(-1)^n}{2n-1} \bar{\chi}(2n-1)\,G(\chi),$$

by (2.1).

We integrate f over the same rectangle C_N as in the proof of Theorem 3.2. By an argument similar to that in that proof, we find that

$$(3.13)\qquad I_N \equiv \frac{1}{2\pi i} \int_{C_N} f(z)\,dz = o(1),$$

as N tends to ∞. Hence, applying the residue theorem to I_N, using (3.11) and (3.12), letting N tend to ∞, and employing (3.13), we find that

$$0 = 2\pi S_{41} + G(\chi) \sum_{n=-\infty}^{\infty} \frac{(-1)^n \bar{\chi}(2n-1)}{2n-1},$$

from whence Theorem 3.7 follows.

A proof of Theorem 3.7 using the character Poisson formula may be found in [7, section 4].

COROLLARY 3.8. If χ is real and even, then $S_{41} > 0$.

Additional class number formulas of Dirichlet are immediate consequences of Theorem 3.7.

COROLLARY 3.9. If $4 \nmid d$, then

(3.14) $$S_{41}(\chi_d) = \frac{1}{2} h(-4d), \quad d > 0,$$

$$S_{41}(\chi_{-4d}) = 2h(d), \quad d < 0,$$

(3.15) $$S_{41}(\chi_{8d}) = h(-8d), \quad d > 0,$$

and

(3.16) $$S_{41}(\chi_{-8d}) = h(8d), \quad d < 0.$$

COROLLARY 3.10. If $p \equiv 1 \pmod 8$, then $h(-4p) \equiv 0 \pmod 4$; if $p \equiv 5 \pmod 8$, then $h(-4p) \equiv 2 \pmod 4$. If p is odd, then $h(-8p)$ is even.

Proof. The number of summands in $S_{41}(\chi_p)$ is even if $p \equiv 1 \pmod 8$ and odd if $p \equiv 5 \pmod 8$. Thus, the congruences for $h(-4p)$ readily follow from (3.14). For all odd primes p, $S_{41}(\chi_{8p})$ has $2p$ terms and, thus, $p - 1$ non-zero summands. Hence, $S_{41}(\chi_{8p})$ is even, and (3.15) and (3.16) show that $h(-8p)$ is even.

The congruences for $h(-4p)$ in Corollary 3.10 appear to have been first stated by Lerch [45, p. 224], although they were, no doubt, known to Dirichlet. For other proofs of the congruences in Corollary 3.10, for equivalent formulations, and for some refinements, see the papers of Brown [10], [11], [14], Hasse [32], [33], [34], and Barrucand and Cohn [4].

4. Sums over intervals of length $k/3$.

THEOREM 4.1. If χ is even and $\chi_{3k}(n) = \left(\frac{n}{3}\right)\chi(n)$, then

$$S_{31} = \frac{3^{1/2}\,G(\chi)}{2\pi} L(1, \bar{\chi}_{3k}); \tag{4.1}$$

if χ is odd, then

$$S_{31} = \frac{G(\chi)}{2\pi i}\{3 - \bar{\chi}(3)\} L(1, \bar{\chi}). \tag{4.2}$$

Proof. First, suppose that χ is even. Let

$$f(x) = \begin{cases} 1, & 0 \leqslant x < 2\pi/3, \\ 1/2, & x = 2\pi/3, \\ 0, & 2\pi/3 < x \leqslant \pi, \end{cases}$$

be an even function with period 2π. Then, by an elementary calculation,

$$f(x) = \frac{2}{3} + \frac{3^{1/2}}{\pi}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{\cos(nx)}{n} \qquad (-\infty < x < \infty). \tag{4.3}$$

Now, multiply both sides of (2.1) by $3^{1/2}\left(\frac{n}{3}\right)/(\pi n)$ and sum on n, $1 \leqq n < \infty$. With the use of (4.3), we obtain

$$\begin{aligned} 2S_{31} &= \sum_{j=1}^{k-1} \chi(j)\{f(2\pi j/k) - 2/3\} \\ &= \frac{3^{1/2}}{\pi} G(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\left(\frac{n}{3}\right)\frac{1}{n} = \frac{3^{1/2}}{\pi} G(\chi) L(1, \bar{\chi}_{3k}), \end{aligned}$$

which completes the proof of (4.1).

For variety, we shall prove (4.2) by contour integration. Of course, the method of Fourier series used above works equally well here.

Let

$$f(z) = \frac{\pi F(z, \chi)}{z \sin \pi(z + 1/3)},$$

where

$$\begin{aligned} F(z, \chi) = 2i &\sum_{0<j<k/3} \chi(j) \sin(\pi z + \pi/3 - 6\pi j z/k) \\ &+ e^{-3\pi i z}\sum_{k/3<j<2k/3} \chi(j)\, e^{6\pi i j z/k}. \end{aligned}$$

Observe that

$$R(f, 0) = \frac{\pi F(0, \chi)}{\sin(\pi/3)} = 2\pi i S_{31} \tag{4.4}$$

and that

$$R(f, n-1/3) = \frac{3(-1)^n}{3n-1} F(n-1/3, \chi) \tag{4.5}$$

$$= -\frac{3}{3n-1} G(3n-1, \chi) = -\frac{3}{3n-1} \bar{\chi}(3n-1) G(\chi),$$

by (2.1), where $-\infty < n < \infty$.

We integrate f over the same rectangle C_N as in the proof of Theorem 3.2. The estimate (3.8) is obtained by the same type of argument as in that proof. Applying the residue theorem, employing (4.4) and (4.5), and letting N tend to ∞, we deduce that

$$0 = 2\pi i\, S_{31} - 3G(\chi) \sum_{n=-\infty}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1}$$

$$= 2\pi i\, S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n-1)}{3n-1} + \sum_{n=0}^{\infty} \frac{\bar{\chi}(3n+1)}{3n+1} \right\}$$

$$= 2\pi i\, S_{31} - 3G(\chi) \left\{ \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} - \sum_{n=1}^{\infty} \frac{\bar{\chi}(3n)}{3n} \right\},$$

from which (4.2) readily follows.

Corollary 4.2. For any real primitive character χ with $k > 3$, $S_{31} > 0$.

Corollary 4.3. If $d > 0$ and $3 \nmid d$, then

$$S_{31}(\chi_d) = \frac{1}{2} h(-3d); \tag{4.6}$$

if $d < 0$, then

$$S_{31}(\chi_{-d}) = \frac{1}{2} \left\{ 3 - \left(\frac{d}{3}\right) \right\} h(d). \tag{4.7}$$

Corollary 4.4. Let $p > 3$. If $p \equiv 1 \pmod{12}$, then $h(-3p) \equiv 0 \pmod 4$, while if $p \equiv 5 \pmod{12}$, then $h(-3p) \equiv 2 \pmod 4$. If $p \equiv 3 \pmod 4$, then $h(-12p) \equiv 0 \pmod 4$. For any odd prime p, $h(-24p) \equiv 0 \pmod 4$.

Proof. Let $p = 6m + j$, where $j = 1$ or 5 and m is a non-negative integer. The number of summands in $S_{31}(\chi_p)$ is thus $2m + [j/3]$. The two congruences for $h(-3p)$ are then consequences of (4.6). The number of summands in $S_{31}(\chi_{4p})$ is $8m + [4j/3]$. If $p \equiv 7 \pmod{12}$, the number of non-zero summands is $4m$; if $p \equiv 11 \pmod{12}$, the number of non-zero summands is $4m + 2$. In either case, the number of non-zero summands is even, and so it follows from (4.6) that $h(-12p) \equiv 0 \pmod 4$ when $p \equiv 3 \pmod 4$. Lastly, the number of summands in $S_{31}(\chi_{8p})$ is $16m + [8j/3]$. If $j = 1$, there are $8m$ non-zero summands; if $j = 5$, there are $8m + 6$ non-zero summands. In either case, $S_{31}(\chi_{8p})$ is even, and we deduce from (4.6) that $h(-24p) \equiv 0 \pmod 4$.

COROLLARY 4.5. Let p and q be distinct primes with $p, q > 3$ and $p \equiv q \pmod 4$. Then $h(-3pq) \equiv 0 \pmod 4$.

Proof. Let $p = 6m + j$ and $q = 6m' + j'$, where $j, j' = 1$ or 5 and m and m' are non-negative integers. The number of summands in $S_{31}(\chi_{pq})$ is $[pq/3]$, and we observe that $[pq/3] \equiv [jj'/3] \pmod 2$. Of these summands, $[q/3] = 2m' + [j'/3]$ are multiples of p, and $[p/3] = 2m + [j/3]$ are multiples of q. Thus,

$$S_{31}(\chi_{pq}) \equiv [jj'/3] - [j'/3] - [j/3] \pmod 2 .$$

By examining all of the possibilities for the pair j, j', we find that $S_{31}(\chi_{pq})$ is always even. The result now follows from (4.6).

It is clear that the same type of argument yields congruences from $h(-12pq)$ and $h(-24pq)$.

The class number formulae (4.6) and (4.7) appear to be due originally to Lerch [44, pp. 402, 408]. Holden [36] has also given a proof of (4.7).

5. SUMS OVER INTERVALS OF LENGTH $k/5$.

THEOREM 5.1. Let χ be odd and let $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$. Then

$$(5.1) \quad S_{51} = \frac{1}{4\pi i} G(\chi) \{ (5 - \bar{\chi}(5)) L(1, \bar{\chi}) - 5^{1/2} L(1, \bar{\chi}_{5k}) \}$$

and

$$(5.2) \quad S_{52} = \frac{1}{2\pi i} 5^{1/2} G(\chi) L(1, \bar{\chi}_{5k}) .$$

Proof. Let

$$f(x) = \begin{cases} 1, & 0 < x < 2\pi/5\,, \\ 1/2, & x = 2\pi/5\,, \\ 0, & 2\pi/5 < x \leqslant \pi\,, \end{cases}$$

be an odd function of period 2π. Calculating the Fourier series of f, we find that, for all x,

$$(5.3) \qquad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} - \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{\sin(5nx)}{n}$$

$$+ \frac{2}{\pi}\cos(\pi/5)\sum_{\substack{n=1\\ n\equiv 2,3 \ (\mathrm{mod}\ 5)}}^{\infty}\frac{\sin(nx)}{n} - \frac{2}{\pi}\cos(2\pi/5)\sum_{\substack{n=1\\ n\equiv 1,4 \ (\mathrm{mod}\ 5)}}^{\infty}\frac{\sin(nx)}{n}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)}{n} - \frac{2}{5\pi}\sum_{n=1}^{\infty}\frac{\sin(5nx)}{n}$$

$$+ \frac{1}{\pi}\cos(\pi/5)\sum_{n=1}^{\infty}\left\{1-\left(\frac{n}{5}\right)\right\}\frac{\sin(nx)}{n}$$

$$- \frac{1}{\pi}\cos(2\pi/5)\sum_{n=1}^{\infty}\left\{1+\left(\frac{n}{5}\right)\right\}\frac{\sin(nx)}{n}$$

$$+ \frac{1}{5\pi}\{\cos(2\pi/5) - \cos(\pi/5)\}\sum_{n=1}^{\infty}\frac{\sin(5nx)}{n}$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left\{5 - 5^{1/2}\left(\frac{n}{5}\right)\right\}\frac{\sin(nx)}{n} - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{\sin(5n\ x)}{n}\,,$$

since $\cos(\pi/5) = (5^{1/2}+1)/4$ and $\cos(2\pi/5) = (5^{1/2}-1)/4$.

Now, multiply both sides of (2.1) by $\left\{5 - 5^{1/2}\left(\frac{n}{5}\right)\right\}/(2\pi n)$ and sum on n, $1 \leqq n < \infty$. Next, replace n by $5n$ in (2.1) and then multiply both sides of (2.1) by $-1/(2\pi n)$ and sum on n, $1 \leqq n < \infty$. Adding the resulting two equations and using (5.3), we get

$$2i\,S_{51} = i\sum_{j=1}^{k-1}\chi(j)f(2\pi j/k)$$

$$= \frac{G(\chi)}{2\pi}\left\{\sum_{n=1}^{\infty}\left\{5 - 5^{1/2}\left(\frac{n}{5}\right)\right\}\frac{\bar\chi(n)}{n} - \sum_{n=1}^{\infty}\frac{\bar\chi(5n)}{n}\right\}$$

$$= \frac{G(\chi)}{2\pi}\{\,5L(1,\bar\chi) - 5^{1/2}L(1,\bar\chi_{5k}) - \bar\chi(5)\,L(1,\bar\chi)\,\}\,,$$

from which (5.1) follows immediately.

The proof of (5.2) is similar. In this case, we let

$$f(x) = \begin{cases} 0, & 0 \leqslant x < 2\pi/5,\ 4\pi/5 < x \leqslant \pi, \\ 1/2, & x = 2\pi/5,\ 4\pi/5, \\ 1, & 2\pi/5 < x < 4\pi/5, \end{cases}$$

be an odd function with period 2π. The Fourier series of f is given by

$$f(x) = \frac{5^{1/2}}{\pi} \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{\sin(nx)}{n} \qquad (-\infty < x < \infty).$$

We then proceed in the same fashion as above.

COROLLARY 5.2. If χ is real and odd, then $S_{52} > 0$.

COROLLARY 5.3. If $d < 0$ and $5 \nmid d$, then

(5.4) $$S_{51}(\chi_{-d}) = \frac{1}{4}\left\{5 - \left(\frac{d}{5}\right)\right\} h(d) - \frac{1}{4} h(5d)$$

and

(5.5) $$S_{52}(\chi_{-d}) = \frac{1}{2} h(5d)\,.$$

Formula (5.5) is due to Lerch [44, p. 407]. By combining (5.4) and (5.5), we can derive a formula for $h(d)$ which is also due to Lerch [44, p. 404].

COROLLARY 5.4. If $p \neq 5$, we have the following consequences:

(5.6) $h(-5p) \equiv 0 \pmod 8$, if $p \equiv 19 \pmod{20}$,

(5.7) $h(-5p) \equiv 4 \pmod 8$, if $p \equiv 11 \pmod{20}$,

(5.8) $h(-5p) \equiv 2h(-p) \pmod 8$, if $p \equiv 7 \pmod{20}$,

(5.9) $h(-5p) \equiv 4 + 2h(-p) \pmod 8$, if $p \equiv 3 \pmod{20}$,

(5.10) $h(-20p) \equiv 0 \pmod 8$, if $p \equiv 1,9 \pmod{20}$ or if $p \equiv 13,37 \pmod{40}$,

(5.11) $h(-20p) \equiv 4 \pmod 8$, if $p \equiv 17,33 \pmod{40}$,

(5.12) $h(-40p) \equiv 4 \pmod 8$, if $p \equiv 2,3 \pmod 5$,

and

(5.13) $h(-40p) \equiv 2h(-8p) \pmod 8$, if $p \equiv 1,4 \pmod 5$.

Proof. If $p \equiv j \pmod{10}$, $1 \leqq j \leqq 9$, then $S_{51}(\chi_p) \equiv [j/5] \pmod 2$. With the use of (5.4) and the above, and recalling that $h(-p)$ is odd, we deduce (5.6)-(5.9).

If $p \equiv j \pmod 5$, $1 \leqq j \leqq 4$, the number of non-zero summands in $S_{51}(\chi_{4p})$ is even if $j = 1$ or 4 and is odd if $j = 2$ or 3. Using also Corollary 3.10, we readily deduce (5.10) and (5.11) from (5.4).

If $p \equiv j \pmod 5$, $1 \leqq j \leqq 4$, the number of non-zero summands in $S_{51}(\chi_{8p})$ is even if $j = 1$ or 4 and is odd if $j = 2$ or 3. Using also the fact that $h(-8p)$ is even, we may deduce (5.12) and (5.13) from (5.4).

COROLLARY 5.5. Let p and q be primes with $p, q \neq 5$ and with $p \equiv q + 2 \pmod 4$. Then

$$h(-5pq) \equiv 0 \pmod 8, \text{ if } p \equiv 1,9 \pmod{20} \text{ and } q \equiv 11,19 \pmod{20},$$
$$h(-5pq) \equiv 4 \pmod 8, \text{ if } p \equiv 13,17 \pmod{20} \text{ and } q \equiv 3,7 \pmod{20},$$

and

$$h(-5pq) \equiv 2h(-pq), \text{ if } p \equiv 1,9 \pmod{20} \text{ and } q \equiv 3,7 \pmod{20},$$
$$\text{or if } p \equiv 13,17 \pmod{20} \text{ and } q = 11,19 \pmod{20}.$$

Of course, the same congruences for $h(-5pq)$ hold if the congruences for p and q are interchanged.

Proof. Let $p \equiv j \pmod{10}$ and $q \equiv j' \pmod{10}$, where $1 \leqq j, j' \leqq 9$. Observe that $S_{51}(\chi_{pq})$ contains $[pq/5]$ terms of which $[q/5]$ are multiples of p and $[p/5]$ are multiples of q. From (5.4), we then find that

$$4([jj'/5] - [j/5] - [j'/5])$$
$$\equiv \left\{5 - \left(\frac{5}{p}\right)\left(\frac{5}{q}\right)\right\} h(-pq) - h(-5pq) \pmod 8 .$$

Since $h(-pq)$ is even, each of the desired congruences readily follows.

In the case that χ is even, we can state a theorem analogous to Theorem 5.1. However, the L-functions in the representations of S_{51} and S_{52} involve quartic characters. For example,

$$(5.14) \qquad S_{51} = \frac{G(\chi)}{\pi}\left\{\sin(2\pi/5) \sum_{\substack{n=1 \\ n \equiv 1,4 \pmod 5}}^{\infty} \frac{(-1)^{n+1}\bar{\chi}(n)}{n} + \sin(\pi/5) \sum_{\substack{n=1 \\ n \equiv 2,3 \pmod 5}}^{\infty} \frac{(-1)^n \bar{\chi}(n)}{n}\right\};$$

the series on the right side of (5.14) may be written in terms of L-functions of quartic characters. Thus, we are unable to derive any positivity results for character sums.

6. Sums over intervals of length $k/6$.

Theorem 6.1. Let χ be even and let $\chi_{3k}(n) = \left(\frac{n}{3}\right)\chi(n)$. Then

$$S_{61} = \frac{3^{1/2}G(\chi)}{2\pi}\{1 + \bar{\chi}(2)\}L(1, \bar{\chi}_{3k}), \tag{6.1}$$

$$S_{62} = -\frac{3^{1/2}G(\chi)}{2\pi}\bar{\chi}(2)L(1, \bar{\chi}_{3k}), \tag{6.2}$$

and

$$S_{63} = -\frac{3^{1/2}G(\chi)}{2\pi}L(1, \bar{\chi}_{3k}). \tag{6.3}$$

Let χ be odd. Then

$$S_{61} = \frac{G(\chi)}{2\pi i}\{1 + \bar{\chi}(2) + \bar{\chi}(3) - \bar{\chi}(6)\}L(1, \bar{\chi}), \tag{6.4}$$

$$S_{62} = \frac{G(\chi)}{2\pi i}\{2 - \bar{\chi}(2) - 2\bar{\chi}(3) + \bar{\chi}(6)\}L(1, \bar{\chi}), \tag{6.5}$$

and

$$S_{63} = \frac{G(\chi)}{2\pi i}\{1 - 2\bar{\chi}(2) + \bar{\chi}(3)\}L(1, \bar{\chi}). \tag{6.6}$$

We shall not give a proof of Theorem 6.1, because all of the formulas may be deduced from Theorems 3.2 and 4.1 and elementary considerations.

Corollary 6.2. If $d > 0$, we have

$S_{61} > 0$, if d is even, or if $\chi(2) = 1$;

$S_{61} = 0$, if $\chi(2) = -1$;

$S_{62} > 0$, if $\chi(2) = -1$;

$S_{62} = 0$, if d is even;

$S_{62} < 0$, if $\chi(2) = 1$;

$S_{63} < 0$, for all d;

$$S_{61} = -S_{63}, \text{ if } d \text{ is even};$$

$$S_{61} = -2S_{62} = -2S_{63}, \text{ if } \chi(2) = 1;$$

and

$$S_{62} = -S_{63}, \text{ if } \chi(2) = -1.$$

If $d < 0$, we have

$S_{61} > 0$, if d is even and $\chi(3) = 1$ or 0, or if $\chi(2) = 1$, or if $\chi(2) = -\chi(3) = -1$;

$S_{61} = 0$, if d is even and $\chi(3) = -1$, or if $\chi(3) = 0$ and $\chi(2) = -1$;

$S_{61} < 0$, if $\chi(2) = \chi(3) = -1$;

$S_{62} > 0$, if d is even and $\chi(3) = -1$, or if $\chi(3) \neq 1$;

$S_{62} = 0$, if $\chi(3) = 1$;

$S_{63} > 0$, if d is even and $\chi(3) \neq -1$, or if $\chi(2) = -1$;

$S_{63} = 0$, if d is even and $\chi(3) = -1$, or if $\chi(2) = \chi(3) = 1$;

and

$S_{63} < 0$, if $\chi(2) = 1$ and $\chi(3) \neq 1$.

We remark here that the results $S_{6i} = 0$, $i = 1, 2, 3$, in Corollary 6.2 may be proven in a completely elementary manner. As an illustration, we prove that $S_{61} = 0$ if χ is even and $\chi(2) = -1$. (The following argument was supplied to the author by Thomas Cusick, Ronald J. Evans, and the author's students in a graduate course in number theory.) Since χ is even and $\chi(2) = -1$, we have

$$\begin{aligned}\sum_{k/3<n<k/2} \chi(n) &= \sum_{\substack{k/3<n<k/2 \\ n \text{ even}}} \chi(n) + \sum_{\substack{k/3<n<k/2 \\ n \text{ odd}}} \chi(n) \\ &= \chi(2) \sum_{k/6<n<k/4} \chi(n) + \sum_{k/4<n<k/3} \chi(k-2n) \\ &= -\sum_{k/6<n<k/3} \chi(n).\end{aligned}$$

As $S_{21} = 0$, it follows from the above that $S_{61} = 0$.

In the case that $\chi(n)$ is the Legendre symbol, the equalities of Corollary 6.2 were derived by Johnson and Mitchell [41].

Of course, using (2.4), we may convert (6.1)-(6.6) into formulas involving class numbers. Since no new, additional congruences for class numbers may be derived from these formulas, we shall not write them down. The class number formula for $S_{61}(\chi_{-d})$ is due to Lerch [44, p. 403], and those for $S_{62}(\chi_d)$ and $S_{63}(\chi_d)$ are also due to Lerch [44, p. 414]. In the terminology of class numbers, Holden [36] has established (6.4)-(6.6) in the associated special cases. Some results related to (6.1)-(6.3) were also found by Holden [39].

7. Sums over intervals of length $k/8$.

THEOREM 7.1. Let χ be even, let $\chi_{4k} = \chi_4\chi$, and let $\chi_{8k} = \chi_4\chi_8\chi$. Then

$$(7.1)\quad S_{81} = \frac{G(\chi)}{2\pi}\{\bar\chi(2)\,L(1,\bar\chi_{4k}) + 2^{1/2}\,L(1,\bar\chi_{8k})\},$$

$$S_{82} = \frac{G(\chi)}{2\pi}\{[2-\bar\chi(2)]\,L(1,\bar\chi_{4k}) - 2^{1/2}\,L(1,\bar\chi_{8k})\},$$

$$S_{83} = \frac{G(\chi)}{2\pi}\{-[2+\bar\chi(2)]\,L(1,\bar\chi_{4k}) + 2^{1/2}\,L(1,\bar\chi_{8k})\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi}\{\bar\chi(2)\,L(1,\bar\chi_{4k}) - 2^{1/2}\,L(1,\bar\chi_{8k})\}.$$

Let χ be odd and let $\chi_{8k} = \chi_8\chi$. Then

$$(7.2)\quad S_{81} = \frac{G(\chi)}{2\pi i}\left\{\left[2+\frac{1}{2}\bar\chi(4)\{1-\bar\chi(2)\}\right]L(1,\bar\chi) - 2^{1/2}\,L(1,\bar\chi_{8k})\right\},$$

$$S_{82} = \frac{G(\chi)}{2\pi i}\left\{\bar\chi(2)\left[1-\frac{3}{2}\bar\chi(2)+\frac{1}{2}\bar\chi(4)\right]L(1,\bar\chi) + 2^{1/2}\,L(1,\bar\chi_{8k})\right\},$$

$$S_{83} = \frac{G(\chi)}{2\pi i}\left\{\bar\chi(2)\left[-1+\frac{3}{2}\bar\chi(2)-\frac{1}{2}\bar\chi(4)\right]L(1,\bar\chi) + 2^{1/2}\,L(1,\bar\chi_{8k})\right\},$$

and

$$S_{84} = \frac{G(\chi)}{2\pi i}\left\{\left[2-\frac{1}{2}\bar\chi(4)\right][1-\bar\chi(2)]\,L(1,\bar\chi) - 2^{1/2}\,L(1,\bar\chi_{8k})\right\}.$$

We need only prove (7.1) and (7.2), for the remaining formulae can then be deduced from (7.1), (7.2), Theorem 3.2, Theorem 3.7, and elementary considerations. Since the proofs are similar to those in previous sections, we omit them. For the same reasons, proofs in sections 8-11 will not be given.

COROLLARY 7.2. If $d > 0$, we have

$$S_{81} > 0, \text{ if } \chi(2) = 1 \text{ or } 0;$$

$$S_{84} < 0, \text{ if } \chi(2) = -1 \text{ or } 0;$$

$$|S_{84}| < S_{81}, \text{ if } \chi(2) = 1;$$

$$|S_{81}| < -S_{84}, \text{ if } \chi(2) = -1;$$

$$S_{81} > S_{83}, \text{ if } \chi(2) = 1;$$

$$S_{82} > -S_{83}, \text{ if } \chi(2) = -1;$$

$$S_{82} < -S_{83}, \text{ if } \chi(2) = 1;$$

$$S_{82} > S_{84}, \text{ if } \chi(2) = -1;$$

$$S_{81} = S_{83}, \text{ if } \chi(2) = -1;$$

and

$$S_{82} = S_{84}, \text{ if } \chi(2) = 1.$$

If $d < 0$, we have

$$S_{82} > 0, \text{ if } \chi(2) = 1 \text{ or } 0;$$

$$S_{83} > 0;$$

$$S_{84} < 0, \text{ if } \chi(2) = 1;$$

$$|S_{82}| < S_{83}, \text{ if } \chi(2) = -1;$$

$$S_{81} > -S_{83};$$

$$S_{81} = -S_{82} = S_{84}, \text{ if } \chi(2) = -1;$$

and

$$S_{82} = S_{83} = -S_{84}, \text{ if } \chi(2) = 1.$$

Theorem 7.1 yields 8 formulae for class numbers. We shall list just those that we need to derive congruences.

COROLLARY 7.3. Let d be odd. If $d > 0$, then

$$S_{81}(\chi_d) = \frac{1}{4}\left(\frac{d}{2}\right) h(-4d) + \frac{1}{4} h(-8d) \tag{7.3}$$

and

$$S_{84}(\chi_d) = \frac{1}{4}\left(\frac{d}{2}\right) h(-4d) - \frac{1}{4} h(-8d). \tag{7.4}$$

If $d < 0$, then

$$(7.5)\qquad S_{81}(\chi_{-d}) = \frac{1}{4}\left\{5 - \left(\frac{d}{2}\right)\right\} h(d) - \frac{1}{4} h(8d)\,,$$

$$(7.6)\qquad S_{83}(\chi_{-d}) = \frac{3}{4}\left\{1 - \left(\frac{d}{2}\right)\right\} h(d) + \frac{1}{4} h(8d)\,,$$

and

$$(7.7)\qquad S_{84}(\chi_{-d}) = \frac{3}{4}\left\{1 - \left(\frac{d}{2}\right)\right\} h(d) - \frac{1}{4} h(8d)\,.$$

Corollary 7.4. We have

$$h(-8p) \equiv h(-4p) \pmod 8, \text{ if } p \equiv 1,5 \pmod{16},$$
$$h(-8p) \equiv 4 + h(-4p) \pmod 8, \text{ if } p \equiv 9,13 \pmod{16},$$
$$h(-8p) \equiv 0 \pmod 8, \text{ if } p \equiv 15 \pmod{16},$$
$$h(-8p) \equiv 4 \pmod 8, \text{ if } p \equiv 7 \pmod{16},$$
$$h(-8p) \equiv 2h(-p) \pmod 8, \text{ if } p \equiv 11 \pmod{16},$$

and

$$h(-8p) \equiv -2h(-p) \pmod 8, \text{ if } p \equiv 3 \pmod{16}.$$

Proof. If $p \equiv j \pmod{16}$, $1 \leqq j \leqq 15$, then

$$(7.8)\qquad S_{81} \equiv [j/8] \pmod 2.$$

Let $p \equiv 1 \pmod 4$. Then the first two congruences follow from (7.3), (7.8), and Corollary 3.10. Let $p \equiv 3 \pmod 4$. Then the latter four congruences follow from (7.5), (7.8), and the fact that $h(-p)$ is odd.

Corollary 7.5. We have

$$h(-8p) \equiv 0 \pmod 4, \text{ if } p \equiv 1,7 \pmod 8$$

and

$$h(-8p) \equiv 2 \pmod 4, \text{ if } p \equiv 3,5 \pmod 8.$$

Proof. Let $p \equiv 1 \pmod 4$, and suppose that $p \equiv j \pmod{16}$, $1 \leqq j \leqq 15$. Then

$$(7.9)\qquad S_{81} - S_{84} \equiv [j/8] - [j/2] + [3j/8] \pmod 2.$$

The congruences for $p \equiv 1 \pmod 4$ follow from (7.3), (7.4), and (7.9).

Let $p \equiv 3 \pmod 4$, and suppose that $p \equiv j \pmod 8$, $1 \leqq j \leqq 7$. Then

$$(7.10)\qquad S_{83} - S_{84} \equiv -[j/2] - [j/4] \pmod 2.$$

The congruences for $p \equiv 3 \pmod 4$ follow from (7.6), (7.7), and (7.10).

COROLLARY 7.6. We have

$$h(-40p) \equiv 0 \pmod 8, \text{ if } p \equiv 1, 9, 31, 39 \pmod{40}$$

and

$$h(-40p) \equiv 4 \pmod 8, \text{ if } p \equiv 11, 19, 21, 29 \pmod{40}.$$

Proof. The congruences follow from (5.13) and Corollary 7.5.

The character sums of this section were studied in great detail from an elementary viewpoint by Osborn [50] and Glaisher [27], [28], [29]. Some of the class number formulas in this section can be traced back to Gauss [26] with the proofs given by Dedekind [21]. The formulas

$$\frac{1}{2} h(-8d) = S_{81}(\chi_d) - S_{84}(\chi_d) \tag{7.11}$$

and

$$\frac{1}{2} h(8d) = S_{82}(\chi_{-d}) + S_{83}(\chi_{-d}) \tag{7.12}$$

are due to Dirichlet [23]. Proofs of (7.11) and (7.12) were also given by Lerch [44, pp. 407, 409]. Pepin [51], Hurwitz [40], Glaisher [29], Holden [39], Karpinski [42], and Rédei [57] have also derived class number formulas in terms of S_{8i}, $1 \leqq i \leqq 4$.

For $p \equiv 1 \pmod 8$, Corollary 7.5 was first established by Lerch [45, p. 225]. Brown [14] has proven Corollary 7.5 and all the congruences of Corollary 7.4 involving a single class number. He has also pointed out (personal communication) that the remaining congruences of Corollary 7.4 may be deduced from his work [14] and a paper of Hasse [35]. The latter author [32] has also proved Corollary 7.5 for $p \equiv 7 \pmod 8$. As indicated in the Introduction, Corollaries 7.4 and 7.5 have also been proven by Pizer [52]. The special case of Corollary 7.5 when $p \equiv 19 \pmod{24}$ was brought into prominence by Stark [59]. See also [13].

8. SUMS OVER INTERVALS OF LENGTH $k/10$.

As with intervals of length $k/5$, we are able to establish theorems about positive sums for odd χ only.

THEOREM 8.1. Let χ be odd and put $\chi_{5k}(n) = \left(\frac{n}{5}\right) \chi(n)$. Then

$$S_{10,1} = \frac{G(\chi)}{4\pi i} \{[4+\{1-\bar{\chi}(2)\}\{\bar{\chi}(5)-1\}] L(1,\bar{\chi}) - 5^{1/2}[1+\bar{\chi}(2)] L(1,\bar{\chi}_{5k})\},$$

$$S_{10,2} = \frac{G(\chi)}{4\pi i} \{[2-\bar{\chi}(2)][1-\bar{\chi}(5)] L(1,\bar{\chi}) + 5^{1/2}\bar{\chi}(2) L(1,\bar{\chi}_{5k})\},$$

$$S_{10,3} = \frac{G(\chi)}{4\pi i} \{[2-\bar{\chi}(2)][\bar{\chi}(5)-1] L(1,\bar{\chi}) + 5^{1/2}[2+\bar{\chi}(2)] L(1,\bar{\chi}_{5k})\},$$

$$S_{10,4} = \frac{G(\chi)}{4\pi i} \{[2-\bar{\chi}(2)][1-\bar{\chi}(5)] L(1,\bar{\chi}) - 5^{1/2}\bar{\chi}(2) L(1,\bar{\chi}_{5k})\},$$

and

$$S_{10,5} = \frac{G(\chi)}{4\pi i} \{[3-4\bar{\chi}(2)+\bar{\chi}(5)] L(1,\bar{\chi}) - 5^{1/2} L(1,\bar{\chi}_{5k})\}.$$

Corollary 8.2. If $d < 0$, we have

$S_{10,1} > 0$, if $\chi(2) = -1$ and $\chi(5) \neq -1$;

$S_{10,1} = 0$, if $\chi(2) = -1$ and $\chi(5) = -1$;

$S_{10,2} > 0$, if $\chi(2) = 1$, or if $\chi(2) = 0$ and $\chi(5) \neq 1$;

$S_{10,2} = 0$, if $\chi(2) = 0$ and $\chi(5) = 1$;

$S_{10,2} < 0$, if $\chi(2) = -1$ and $\chi(5) = 1$;

$S_{10,3} > 0$, if $\chi(5) = 1$;

$S_{10,4} > 0$, if $\chi(2) = -1$, or if $\chi(2) = 0$ and $\chi(5) \neq 1$;

$S_{10,4} = 0$, if $\chi(2) = 0$ and $\chi(5) = 1$;

$S_{10,4} < 0$, if $\chi(2) = \chi(5) = 1$;

and

$S_{10,5} < 0$, if $\chi(2) = 1$.

We shall refrain from writing down any of the class number formulas arising from Theorem 8.1, since no further congruences for class numbers may be deduced. The sums $S_{10,i}$, $1 \leqq i \leqq 5$, appear to have been previously discussed only by Karpinski [42] and by Rédei [57] in connection with class numbers.

9. Sums over intervals of length $k/12$.

THEOREM 8.1. Let χ be even, $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, and $\chi_{4k}(n) = \chi_4(n) \chi(n)$. Then

$$S_{12,1} = \frac{G(\chi)}{2\pi} \{[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} \bar{\chi}(2) [1+\bar{\chi}(2)] L(1, \bar{\chi}_{3k})\},$$

$$S_{12,2} = \frac{G(\chi)}{2\pi} \{-[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} [2+\bar{\chi}(2)-\bar{\chi}(4)] L(1, \bar{\chi}_{3k})\},$$

$$S_{12,3} = \frac{G(\chi)}{2\pi} \{2L(1, \bar{\chi}_{4k}) - 3^{1/2} [1+\bar{\chi}(2)] L(1, \bar{\chi}_{3k})\},$$

$$S_{12,4} = \frac{G(\chi)}{2\pi} \{-2L(1, \bar{\chi}_{4k}) + 3^{1/2} L(1, \bar{\chi}_{3k})\},$$

$$S_{12,5} = \frac{G(\chi)}{2\pi} \{[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) - \frac{1}{2} 3^{1/2} [2+\bar{\chi}(2)+\bar{\chi}(4)] L(1, \bar{\chi}_{3k})\},$$

and

$$S_{12,6} = \frac{G(\chi)}{2\pi} \{-[1+\bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{2} 3^{1/2} \bar{\chi}(2) [1+\bar{\chi}(2)] L(1, \bar{\chi}_{3k})\}.$$

Let χ be odd and let $\chi_{12k}(n) = \left(\frac{n}{3}\right) \chi_4(n) \chi(n)$. Then

$$(9.1) \quad S_{12,1} = \frac{G(\chi)}{2\pi i} \left\{\frac{1}{2} [4-\bar{\chi}(2)\{1-\bar{\chi}(2)\}\{1-\bar{\chi}(3)\}] L(1, \bar{\chi}) - 3^{1/2} L(1, \bar{\chi}_{12k})\right\},$$

$$S_{12,2} = \frac{G(\chi)}{2\pi i} \left\{ \left[\frac{1}{2} \bar{\chi}(2) - 1 \right] [1 - \bar{\chi}(2)] [1 - \bar{\chi}(3)] L(1, \bar{\chi}) + 3^{1/2} L(1, \bar{\chi}_{12k}) \right\},$$

$$S_{12,3} = \frac{G(\chi)}{2\pi i} [1 - \bar{\chi}(2)] [1 + \bar{\chi}(2) - \bar{\chi}(3)] L(1, \bar{\chi}),$$

$$S_{12,4} = \frac{G(\chi)}{2\pi i} \{ \bar{\chi}(2) [\bar{\chi}(2) - 1] + 1 - \bar{\chi}(3) \} L(1, \bar{\chi}),$$

$$S_{12,5} = \frac{G(\chi)}{2\pi i} \left\{ \frac{1}{2} [\bar{\chi}(3) - 1] [2 + \bar{\chi}(2) - \bar{\chi}(4)] L(1, \bar{\chi}) + 3^{1/2} L(1, \bar{\chi}_{12k}) \right\},$$

and

$$S_{12,6} = \frac{G(\chi)}{2\pi i} \left\{ \frac{1}{2} [1 - \bar{\chi}(2)] [4 + \bar{\chi}(2) \{ 1 - \bar{\chi}(3) \}] L(1, \bar{\chi}) - 3^{1/2} L(1, \bar{\chi}_{12k}) \right\}.$$

COROLLARY 9.2. If $d > 0$, we have

$S_{12,1} > 0$, if $\chi(2) = 1$, or if $\chi(3) \neq -1$;

$S_{12,1} = 0$, if $\chi(2) \neq 1$ and $\chi(3) = -1$;

$S_{12,2} > 0$, if $\chi(2) \neq -1$ and $\chi(3) = -1$;

$S_{12,2} = 0$, if $\chi(2) = \chi(3) = -1$;

$S_{12,2} < 0$, if $\chi(2) = -1$ and $\chi(3) \neq -1$;

$S_{12,3} > 0$, if $\chi(2) = -1$;

$S_{12,5} < 0$, if $\chi(3) = -1$;

$S_{12,6} > 0$, if $\chi(2) = 1$ and $\chi(3) = -1$;

$S_{12,6} = 0$, if $\chi(2) \neq 1$ and $\chi(3) = -1$;

and

$S_{12,6} < 0$, if $\chi(2) \neq 1$ and $\chi(3) \neq -1$.

If $d < 0$, we have

$S_{12,2} > 0$, if $\chi(2) = 1$, or if $\chi(3) = 1$;

$$S_{12,3} > 0\,, \quad \text{if} \quad \chi(2) = \chi(3) = -1\,, \text{ or if } \chi(2) = 0 \text{ and } \chi(3) \neq 1\,;$$

$$S_{12,3} = 0\,, \quad \text{if} \quad \chi(2) = 1\,, \text{ or if } \chi(2) = 0 \text{ and } \chi(3) = 1\,, \text{ or if } \chi(2) = -1 \text{ and } \chi(3) = 0\,;$$

$$S_{12,3} < 0\,, \quad \text{if} \quad \chi(2) = -1 \text{ and } \chi(3) = 1\,;$$

$$S_{12,4} > 0\,, \quad \text{if} \quad \chi(2) = -1\,, \text{ or if } \chi(2) \neq -1 \text{ and } \chi(3) \neq 1\,;$$

$$S_{12,4} = 0\,, \quad \text{if} \quad \chi(2) \neq -1 \text{ and } \chi(3) = 1\,;$$

$$S_{12,5} > 0\,, \quad \text{if} \quad \chi(2) = -1\,, \text{ or if } \chi(3) = 1\,;$$

and

$$S_{12,6} < 0\,, \quad \text{if} \quad \chi(2) = 1\,.$$

Corollary 9.3. We have

$$h(-12p) \equiv 0 \pmod 8\,, \quad \text{if} \quad p \equiv 23 \pmod{24}\,,$$

and

$$h(-12p) \equiv 4 \pmod 8\,, \text{ if } p \equiv 7, 11, 19 \pmod{24}\,.$$

Proof. From (9.1) and (2.4),

$$(9.2)\quad S_{12,1}(\chi_p) = \frac{1}{4}\left\{4 + \left[1 - \left(\frac{2}{p}\right)\right]\left[1 - \left(\frac{3}{p}\right)\right]\right\} h(-p) - \frac{1}{4} h(-12p)\,.$$

If $p \equiv j \pmod{24}$, $1 \leqq j \leqq 23$, then

$$(9.3)\qquad S_{12,1}(\chi_p) \equiv [j/12] \pmod 2\,.$$

From (9.2) and (9.3) we deduce that

$$4\,[j/12] \equiv \left\{4 + \left[1 - \left(\frac{2}{p}\right)\right]\left[1 - \left(\frac{3}{p}\right)\right]\right\} h(-p) - h(-12p) \pmod 8\,.$$

The desired congruences now readily follow.

The special case, $p \equiv 19 \pmod{24}$, of Corollary 9.3 was important in Stark's work [59]. Brown [13], [14] has also given proofs of this special case.

Some of the class number formulas arising from Theorem 9.1 were actually stated by Gauss [26] with the proofs given by Dedekind [21]. Several class number formulas involving the sums $S_{12,i}$, $1 \leqq i \leqq 6$, were discovered by Lerch [44, pp. 407, 408, 414], Holden [36], [38], [39], Karpinski [42], and Rédei [57].

10. Sums over intervals of length $k/16$

Although $S_{16,i}$, $1 \leqq i \leqq 8$, may be expressed in terms of Dirichlet L-functions at the value 1 by the methods of the previous sections, in each case, L-functions with complex characters arise. Thus, our methods do not enable us to make any conclusions about the sign of $S_{16,i}$, $1 \leqq i \leqq 8$. However, we are able to prove the following result.

Theorem 10.1. Let χ be even and put $\chi_{4k} = \chi_4 \chi$ and $\chi_{8k} = \chi_4 \chi_8 \chi$. Then

$$S_{16,1} + S_{16,8} = \frac{G(\chi)}{2\pi} \{ \bar{\chi}(4) L(1, \bar{\chi}_{4k}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,2} + S_{16,7} = \frac{G(\chi)}{2\pi} \{ [2\bar{\chi}(2) - \bar{\chi}(4)] L(1, \bar{\chi}_{4k}) - 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,3} + S_{16,6} = \frac{G(\chi)}{2\pi} \{ - [2\bar{\chi}(2) + \bar{\chi}(4)] L(1, \bar{\chi}_{4k}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

and

$$S_{16,4} + S_{16,5} = \frac{G(\chi)}{2\pi} \{ \bar{\chi}(4) L(1, \bar{\chi}_{4k}) - 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \}.$$

Let χ be odd and put $\chi_{8k} = \chi_8 \chi$. Then

$$S_{16,1} - S_{16,8} = \frac{G(\chi)}{2\pi i} \{ [2\bar{\chi}(2) + \bar{\chi}(4) + \bar{\chi}(8)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi}) - 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,2} - S_{16,7} = \frac{G(\chi)}{2\pi i} \{ [\bar{\chi}(4) - \bar{\chi}(8)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

$$S_{16,3} - S_{16,6} = \frac{G(\chi)}{2\pi i} \{ [\bar{\chi}(8) - \bar{\chi}(4)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi}) + 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \},$$

and

$$S_{16,4} - S_{16,5} = \frac{G(\chi)}{2\pi i} \{ [2\bar{\chi}(2) - \bar{\chi}(4) - \bar{\chi}(8)] [1 - \frac{1}{2} \bar{\chi}(2)] L(1, \bar{\chi}) - 2^{1/2} \bar{\chi}(2) L(1, \bar{\chi}_{8k}) \} .$$

COROLLARY 10.2. If d is odd and positive, then

$$S_{16,1} + S_{16,8} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$

and

$$S_{16,4} + S_{16,5} > 0 , \quad \text{if} \quad \chi(2) = -1 .$$

If d is odd and negative, then

$$S_{16,2} - S_{16,7} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$

$$S_{16,3} - S_{16,6} > 0 , \quad \text{if} \quad \chi(2) = 1 ,$$

$$S_{16,3} - S_{16,6} < 0 , \quad \text{if} \quad \chi(2) = -1 ,$$

and

$$S_{16,4} - S_{16,5} < 0 , \quad \text{if} \quad \chi(2) = 1 .$$

11. Sums over intervals of length $k/24$.

For intervals of length $k/24$, a complete statement of Theorem 11.1 for both even and odd characters would require 24 formulas. Because of limitations of space, we state just 2 of the formulas for $S_{24,i}(\chi)$, where $1 \leqq i \leqq 12$ and χ is even or odd.

THEOREM 11.1. Let χ be even. Let $\chi_{3k}(n) = \left(\frac{n}{3}\right) \chi(n)$, $\chi_{4k}(n) = \chi_4(n) \chi(n)$, $\chi_{8k}(n) = \chi_4(n) \chi_8(n) \chi(n)$, and $\chi_{24k}(n) = \left(\frac{n}{3}\right) \chi_8(n) \chi(n)$.

Then

$$S_{24,1} = \frac{G(\chi)}{2\pi} \left\{ \frac{1}{2} \bar{\chi}(2) [1 + \bar{\chi}(3)] L(1, \bar{\chi}_{4k}) + \frac{1}{4} 3^{1/2} \bar{\chi}(4) [1 + \bar{\chi}(2)] L(1, \bar{\chi}_{3k}) + 2^{-1/2} [\bar{\chi}(3) - 1] L(1, \bar{\chi}_{8k}) + (3/2)^{1/2} L(1, \bar{\chi}_{24k}) \right\} .$$

Let χ be odd. Put $\chi_{8k}(n) = \chi_8(n)\chi(n)$, $\chi_{12k}(n) = \left(\frac{n}{3}\right)\chi_4(n)\chi(n)$, and $\chi_{24k}(n) = \left(\frac{n}{3}\right)\chi_4(n)\chi_8(n)\chi(n)$. Then

$$S_{24,2} = \frac{G(\chi)}{2\pi i}\left\{\frac{1}{4}\bar{\chi}(2)[2-\bar{\chi}(2)][\bar{\chi}(2)-1][1-\bar{\chi}(3)]L(1,\bar{\chi}) + 2^{-1/2}[1+\bar{\chi}(3)]L(1,\bar{\chi}_{8k}) + 3^{1/2}\left[\frac{1}{2}\bar{\chi}(2)-1\right]L(1,\bar{\chi}_{12k}) + (3/2)^{1/2}L(1,\bar{\chi}_{24k})\right\}.$$

The next result gives the deductions about positive and negative character sums that can be derived from a full statement of Theorem 11.1.

COROLLARY 11.2. If $d > 0$, we have

$S_{24,1} > 0$, if $\chi(2) = \chi(3) = 1$, or if $\chi(2) = 0$ and $\chi(3) = 1$;

$S_{24,3} > 0$, if $\chi(2) = 0$ and $\chi(3) = -1$;

$S_{24,5} > 0$, if $\chi(2) = \chi(3) = -1$;

$S_{24,10} < 0$, if $\chi(2) \neq 1$ and $\chi(3) = -1$, or if $\chi(2) = -1$ and $\chi(3) = 0$;

and

$S_{24,12} < 0$, if $\chi(2) \neq 1$ and $\chi(3) = 1$.

If $d < 0$, we have

$S_{24,4} > 0$, if $\chi(2) = 1$, or if $\chi(2) = 0$ and $\chi(3) = 1$;

$S_{24,6} > 0$, if $\chi(2) = 0$ and $\chi(3) = -1$;

$S_{24,7} > 0$, if $\chi(2) \neq -1$ and $\chi(3) = -1$;

and

$S_{24,9} > 0$, if $\chi(3) = 1$, or if $\chi(2) = -1$ and $\chi(3) = 0$.

We next state just two of the 24 different class number formulas involving $S_{24,i}$ that can be deduced.

COROLLARY 11.3. If $d > 0$, $2 \nmid d$, and $3 \nmid d$, then

$$8S_{24,1}(\chi_d) = \left(\frac{d}{2}\right)\left\{1+\left(\frac{d}{3}\right)\right\}h(-4d) + \left\{1+\left(\frac{d}{2}\right)\right\}h(-3d) + \left\{\left(\frac{d}{3}\right)-1\right\}h(-8d) + h(-24d). \tag{11.1}$$

If $d < 0$, $2 \not| d$, and $3 \not| d$, then

$$(11.2)\quad 8S_{24,2}(\chi_{-d}) = \left\{2\left(\frac{d}{2}\right) - 1\right\}\left\{\left(\frac{d}{2}\right) - 1\right\}\left\{1 - \left(\frac{d}{3}\right)\right\} h(d) + \left\{1 + \left(\frac{d}{3}\right)\right\} h(8d) + \left\{\left(\frac{d}{2}\right) - 2\right\} h(12d) + h(24d).$$

Several congruences for class numbers may be deduced from Corollary 11.3. We remark that the consideration of other class number formulas involving $S_{24,i}$ does not appear to yield further congruences.

COROLLARY 11.4. If $p \equiv 1 \pmod 4$, then

$$(11.3)\quad h(-24p) + 2h(-4p) + 2h(-3p) \equiv 0 \pmod{16}, \quad \text{if } p \equiv 1 \pmod{48},$$

$$(11.4)\quad h(-24p) + 2h(-4p) + 2h(-3p) \equiv 8 \pmod{16}, \quad \text{if } p \equiv 25 \pmod{48},$$

$$(11.5)\quad h(-24p) - 2h(-8p) \equiv 0 \pmod{16}, \text{ if } p \equiv 5 \pmod{48},$$

$$(11.6)\quad h(-24p) - 2h(-8p) \equiv 8 \pmod{16}, \text{ if } p \equiv 29 \pmod{48},$$

$$(11.7)\quad h(-24p) - 2h(-4p) \equiv 0 \pmod{16}, \text{ if } p \equiv 13 \pmod{48},$$

$$(11.8)\quad h(-24p) - 2h(-4p) \equiv 8 \pmod{16}, \text{ if } p \equiv 37 \pmod{48},$$

$$(11.9)\quad h(-24p) - 2h(-8p) + 2h(-3p) \equiv 0 \pmod{16}, \quad \text{if } p \equiv 17 \pmod{48},$$

and

$$(11.10)\quad h(-24p) - 2h(-8p) + 2h(-3p) \equiv 8 \pmod{16}, \quad \text{if } p \equiv 41 \pmod{48}.$$

Proof. Let $p \equiv j \pmod{48}$, $0 < j < 48$. Then by (11.1), we have

$$(11.11)\quad 8[j/24] \equiv \left(\frac{2}{p}\right)\left\{1 + \left(\frac{3}{p}\right)\right\} h(-4p) + \left\{1 + \left(\frac{2}{p}\right)\right\} h(-3p) + \left\{\left(\frac{3}{p}\right) - 1\right\} h(-8p) + h(-24p) \pmod{16}.$$

Congruences (11.3)-(11.10) now follow directly from (11.11) by considering the eight separate cases modulo 48.

COROLLARY 11.5. We have

$$h(-24p) \equiv 0 \pmod 8, \quad \text{if} \quad p \equiv 1 \pmod{24}, \tag{11.12}$$

and

$$h(-24p) \equiv 4 \pmod 8, \quad \text{if} \quad p \equiv 5, 13, 17 \pmod{24}. \tag{11.13}$$

Proof. Congruence (11.12) is a consequence of (11.3), (11.4), Corollary 3.10, and Corollary 4.4. Secondly, for $p \equiv 5 \pmod{24}$, (11.13) follows from (11.5), (11.6), and Corollary 7.5. Thirdly, for $p \equiv 13 \pmod{24}$, (11.13) follows from (11.7), (11.8), and Corollary 3.10. Lastly, for $p \equiv 17 \pmod{24}$, (11.13) follows from (11.9), (11.10), Corollary 4.4, and Corollary 7.5.

COROLLARY 11.6. If $p > 3$ and $p \equiv 3 \pmod 4$, then

$$h(-24p) - h(-12p) \equiv 0 \pmod{16}, \quad \text{if} \quad p \equiv 7 \pmod{48},$$
$$h(-24p) - h(-12p) \equiv 8 \pmod{16}, \quad \text{if} \quad p \equiv 31 \pmod{48},$$
$$h(-24p) - 3h(-12p) + 2h(-8p) \equiv 0 \pmod{16}, \text{ if } p \equiv 11 \pmod{48}$$
$$h(-24p) - 3h(-12p) + 2h(-8p) \equiv 8 \pmod{16}, \text{ if } p \equiv 35 \pmod{48},$$
$$h(-24p) - 3h(-12p) + 4h(-p) \equiv 0 \pmod{16}, \text{ if } p \equiv 19 \pmod{48},$$
$$h(-24p) - 3h(-12p) + 4h(-p) \equiv 8 \pmod{16}, \text{ if } p \equiv 43 \pmod{48},$$
$$h(-24p) - h(-12p) + 2h(-8p) \equiv 8 \pmod{16}, \text{ if } p \equiv 23 \pmod{48},$$

and

$$h(-24p) - h(-12p) + 2h(-8p) \equiv 0 \pmod{16}, \text{ if } p \equiv 47 \pmod{48}.$$

Proof. Let $p \equiv j \pmod{48}$, $0 < j < 48$. Then (11.2) gives

$$\begin{aligned} 8\{[j/12] - [j/24]\} \\ \equiv \left\{2\left(\frac{2}{p}\right) - 1\right\}\left\{\left(\frac{2}{p}\right) - 1\right\}\left\{1 - \left(\frac{3}{p}\right)\right\} h(-p) + \left\{1 + \left(\frac{3}{p}\right)\right\} h(-8p) \\ + \left\{\left(\frac{2}{p}\right) - 2\right\} h(-12p) + h(-24p) \pmod{16}. \end{aligned} \tag{11.14}$$

All of the desired congruences are immediate consequences of (11.14).

COROLLARY 11.7. We have

$$h(-24p) \equiv 0 \pmod 8, \quad \text{if} \quad p \equiv 11, 19, 23 \pmod{24},$$

and

$$h(-24p) \equiv 4 \pmod 8, \quad \text{if} \quad p \equiv 7 \pmod{24}.$$

Proof. The desired congruences follow from Corollaries 7.5, 9.3, and 11.6.

Lerch [44, pp. 409, 410] has derived some class number formulas in terms of the sums $S_{24,i}$, $1 \leq i \leq 12$. Karpinski [42] and Rédei [57] have also established class number relations of this sort.

12. Sums over several intervals of equal length

In this section, it will be convenient to use the following character analogues of the Poisson summation formula [6, Theorem 2.3], [7, equations (4.1), (4.2)]. Let f be continuous and of bounded variation on $[c, d]$. Let χ be a primitive character of modulus k. If χ is even, then

$$(12.1) \quad \sum_{c \leq n \leq d}' \chi(n) f(n) = \frac{2G(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_c^d f(x) \cos(2\pi n x/k)\, dx ;$$

if χ is odd, then

$$(12.2) \quad \sum_{c \leq n \leq d}' \chi(n) f(n) = -\frac{2iG(\chi)}{k} \sum_{n=1}^{\infty} \bar{\chi}(n) \int_c^d f(x) \sin(2\pi n x/k)\, dx .$$

The primes $'$ on the summation signs on the left sides of (12.1) and (12.2) indicate that if c or d is an integer, then the associated summands must be halved.

Throughout the section, it is assumed that χ is a primitive character of modulus k. For each of the theorems below, deductions concerning the signs of the pertinent character sums are trivial. Likewise, the corresponding formulas for class numbers are immediate from (2.4). Thus, none of these obvious corollaries shall be explicitly stated.

THEOREM 12.1. Let χ be even, and let m be any positive integer. Then

$$(12.3) \quad S_{4m,1} + S_{4m,4} + S_{4m,5} + S_{4m,8} + S_{4m,9} + \ldots + S_{4m,4m}$$
$$= \frac{2G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{4k}) .$$

Proof. Apply (12.1) several times with $f(x) \equiv 1$ in each case and with $(c, d) = (0, k/4m)$, $(3k/4m, 5k/4m)$, $(7k/4m, 9k/4m)$, ..., $((4m-1)k/4m, k)$. We then get

$$S_{4m,1} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sin(2\pi n/4m),$$

$$S_{4m,4} + S_{4m,5} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \sin(10\pi n/4m) - \sin(6\pi n/4m) \}$$

$$\vdots$$

$$S_{4m,4m} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ - \sin(2\pi n(4m-1)/4m) \}.$$

Adding the above equations, we find that

$$S_{4m,1} + S_{4m,4} + S_{4m,5} + \ldots + S_{4m,4m} = \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{j=0}^{2m-1} (-1)^j \sin\big(2(2j+1)\pi n/4m\big). \tag{12.4}$$

Now an elementary calculation shows that

$$\sum_{j=0}^{2m-1} (-1)^j \sin\big(2(2j+1)\pi n/4m\big) = \begin{cases} 2m(-1)^{\mu}, & \text{if } n = (2\mu+1)m, \\ 0, & \text{otherwise.} \end{cases} \tag{12.5}$$

Putting (12.5) into (12.4), we conclude that

$$S_{4m,1} + S_{4m,4} + S_{4m,5} + \ldots + S_{4m,4m} = \frac{2G(\chi)}{\pi} \sum_{\mu=0}^{\infty} \frac{(-1)^{\mu} \bar{\chi}((2\mu+1)m)}{2\mu + 1} = \frac{2G(\chi)}{\pi} \bar{\chi}(m) L(1, \bar{\chi}_{4k}),$$

which completes the proof.

Observe that if $m = 1$, Theorem 12.1 reduces to Theorem 3.7. If $m = 2, 3, 4$, and 6, then Theorem 12.1 reduces to results that can be derived from Theorems 7.1, 9.1, 10.1, and 11.1, respectively.

THEOREM 12.2. *Let χ be odd, and let m be a positive integer. If m is odd, then*

$$\sum_{1 \leq j \leq m/2} \left(\frac{m+1}{2} - j \right) S_{m,j} = \frac{G(\chi)}{2\pi i} \{ m - \bar{\chi}(m) \} L(1, \bar{\chi}); \tag{12.6}$$

if m is even, then

$$\sum_{1 \leq j \leq m/2} \left(\frac{m+2}{2} - j \right) S_{m,j} = \frac{G(\chi)}{2\pi i} \{ m + 2 - \bar{\chi}(2) - \bar{\chi}(m) \} L(1, \bar{\chi}). \tag{12.7}$$

Proof. Apply (12.2) several times with $f(x) \equiv 1$ in each case and with $(c, d) = (0, k/m), (k/m, 2k/m), \ldots, (([m/2]-1)\,k/m, [m/2]\,k/m)$. We then get

$$S_{m,1} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \cos\,(2\pi n/m) - 1 \},$$

$$S_{m,2} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \cos\,(4\pi n/m) - \cos\,(2\pi n/m) \},$$

$$\vdots$$

$$S_{m,[m/2]} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \{ \cos\,(2\,[m/2]\,\pi n/m) - \cos\,(2\,\{[m/2]-1\}\,\pi n/m) \}.$$

Multiply the j-th equation above by $[m/2] + 1 - j$, $1 \leqq j \leqq [m/2]$, and add the resulting equations to obtain

$$\sum_{1 \leqq j \leqq m/2} \{[m/2] + 1 - j\}\, S_{m,j} = \frac{iG(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \left\{ -[m/2] + \sum_{j=1}^{[m/2]} \cos\,(2\pi nj/m) \right\}. \tag{12.8}$$

First, suppose that m is odd. Then (12.8) becomes

$$\sum_{1 \leqq j \leqq m/2} \left(\frac{m+1}{2} - j \right) S_{m,j} = \frac{iG(\chi)}{2\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \left\{ -m + \sum_{j=0}^{m-1} \cos\,(2\pi nj/m) \right\}$$
$$= \frac{iG(\chi)}{2\pi} \{ -m + \bar{\chi}(m) \}\, L(1, \bar{\chi}),$$

from which (12.6) follows.

Suppose next that m is even. Then (12.8) becomes

$$\sum_{1 \leqq j \leqq m/2} \left(\frac{m+2}{2} - j \right) S_{m,j}$$
$$= \frac{iG(\chi)}{2\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \left\{ -m - 1 + (-1)^n + \sum_{j=0}^{m-1} \cos\,(2\pi nj/m) \right\}$$
$$= \frac{iG(\chi)}{2\pi} \{ -m - 1 + \bar{\chi}(2) - 1 + \bar{\chi}(m) \}\, L(1, \bar{\chi}),$$

from which (12.7) follows.

We indicate some special cases of the previous theorem. If $m = 2$, (12.7) reduces to Theorem 3.2. If $m = 3$, (12.6) yields Theorem 4.1. If $m = 5$, 6, 8, 10, 12, and 24 in Theorem 12.2, we obtain results deducible from Theorems 5.1, 6.1, 7.1, 8.1, 9.1 and 11.1, respectively.

THEOREM 12.3. Let χ be even and let m be an arbitrary positive integer. Then

$$(12.9)\quad S_{8m,1} - S_{8m,4} - S_{8m,5} + S_{8m,8} + S_{8m,9} - - + + \cdots + S_{8m,8m} = \frac{2^{3/2}\, G(\chi)}{\pi}\, \bar{\chi}(m)\, L(1, \bar{\chi}_{8k}) .$$

Proof. Apply (12.1) several times with $f(x) \equiv 1$ in each instance and with $(c, d) = (0, k/8m)$, $(3k/8m, 5k/8m)$, $(7k/8m, 9k/8m)$, ..., $((8m-1)\, k/8m, k)$. Accordingly, we find that

$$\begin{aligned} S_{8m,1} - S_{8m,4} - S_{8m,5} + S_{8m,8} + S_{8m,9} - - + + \cdots + S_{8m,8m} &= \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{j=0}^{8m-1} \chi_4(j)\, \chi_8(j) \sin(2\pi nj/8m) \\ &= \frac{G(\chi)}{\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{\nu=0}^{7} \chi_4(\nu)\, \chi_8(\nu) \sum_{\mu=0}^{m-1} \sin(2\pi n(8\mu+\nu)/8m) \\ &= \frac{G(\chi)}{\pi}\, \bar{\chi}(m) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \sum_{\nu=0}^{7} \chi_4(\nu)\, \chi_8(\nu) \sin(2\pi n\nu/8) . \end{aligned}$$

The inner sum above is merely $-iG(n, \chi_4\chi_8) = \chi_4(n)\, \chi_8(n)\, 2^{3/2}$, by (2.2). Hence, (12.9) immediately follows.

The special cases with $m = 1$, 2 and 3 of Theorem 12.3 may be deduced from Theorems 7.1, 10.1 and 11.1, respectively.

The proofs of the next four theorems are very similar to the preceding proofs and so will not be given.

THEOREM 12.4. Let χ be odd, and let m be an arbitrary positive integer. Then

$$S_{8m,2} + S_{8m,3} - S_{8m,6} - S_{8m,7} + + - - \cdots - S_{8m,8m-2} - S_{8m,8m-1} = - \frac{i 2^{3/2}\, G(\chi)}{\pi}\, \bar{\chi}(m)\, L(1, \bar{\chi}_{8k}) .$$

The special cases of Theorem 12.4 with $m = 1$, 2 and 3 are consequences of Theorems 7.1, 10.1 and 11.1, respectively.

THEOREM 12.5. Let χ be even, and let m be an arbitrary positive integer. Then

$$\sum_{j=0}^{m-1} S_{3m,3j+2} = -\frac{3^{1/2}G(\chi)}{\pi}\bar{\chi}(m)L(1,\bar{\chi}_{3k}).$$

The instances of Theorem 12.5 with $m = 1, 2, 4$ and 8 are consequences of Theorems 4.1, 6.1, 9.1 and 11.1, respectively.

THEOREM 12.6. Let χ be odd, and let m be an arbitrary positive integer. Then

$$S_{5m,2} - S_{5m,4} + S_{5m,7} - S_{5m,9} + - \cdots + S_{5m,5m-3} - S_{5m,5m-1}$$
$$= -\frac{i5^{1/2}G(\chi)}{\pi}\bar{\chi}(m)L(1,\bar{\chi}_{5k}).$$

The special cases of Theorem 12.6 for $m = 1$ and $m = 2$ follow immediately from Theorems 5.1 and 8.1, respectively.

THEOREM 12.7. Let χ be odd, and let m be an arbitrary positive integer. Then

$$S_{12m,2} + S_{12m,3} + S_{12m,4} + S_{12m,5} - S_{12m,8} - S_{12m,9} - S_{12m,10} - S_{12m,11}$$
$$+ + + + - - - - \cdots - S_{12m,12m-4} - S_{12m,12m-3} - S_{12m,12m-2} - S_{12m,12m-1}$$
$$= -\frac{i(12)^{1/2}G(\chi)}{\pi}\bar{\chi}(m)L(1,\bar{\chi}_{12k}).$$

The special instances of $m = 1$ and $m = 2$ of Theorem 12.7 yield results that are easily deduced from Theorems 9.1 and 11.1, respectively.

The class number formula arising from Theorem 12.1 was first proved by Holden [39]. A less general form of Theorem 12.2 was also established by Holden [36] who in another paper [37] used his result to derive formulas for sums of the Legendre-Jacobi symbol over various residue classes. The special case $m = 1$ of the class number formula deducible from Theorem 12.7 is due to Lerch [44, p. 407]. Otherwise, the results of this section appear to be new.

13. SUMS OF QUADRATIC RESIDUES AND NONRESIDUES

We mentioned in the Introduction the two equivalent formulations of Dirichlet's theorem for primes that are congruent to 3 modulo 4. In this section, we state and prove as many theorems as we can that are of the same nature as (1.3). In the case that $\chi(n)$ is the Legendre symbol, we stated our results in [7, Section 4]. For convenience, we put

$$S_{ji}(\chi, r) = \sum_{(i-1)k/j<n<ik/j} \chi(n) n^r ,$$

where i, j, and r are natural numbers. Again, χ is primitive throughout the section.

THEOREM 13.1. Let χ be even. Then

$$S_{21}(\chi, 1) = -\frac{G(\chi) k}{\pi^2} \left\{1 - \frac{1}{4} \bar{\chi}(2)\right\} L(2, \bar{\chi}) .$$

Proof. In (12.1), put $f(x) = x$, $c = 0$, and $d = k/2$. Integrating by parts, we find that

$$S_{21}(\chi, 1) = \frac{G(\chi) k}{2\pi^2} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n^2} \{\cos(\pi n) - 1\} ,$$

and the desired result readily follows.

COROLLARY 13.2. For any even, real character χ, we have $S_{21}(\chi, 1) < 0$.

In view of Corollary 3.8 and the fact that $S_{21} = 0$ for even χ, Corollary 13.2 is certainly not surprising.

THEOREM 13.3. Let χ be odd. Then

$$S_{21}(\chi, 1) = \frac{iG(\chi) k}{2\pi} \{\bar{\chi}(2) - 1\} L(1, \bar{\chi}) .$$

Proof. In (12.2), put $f(x) = x$, $c = 0$, and $d = k/2$. Thus, upon integrating by parts, we get

$$S_{21}(\chi, 1) = \frac{iG(\chi) k}{2\pi} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)}{n} \cos(\pi n) ,$$

from which the desired result readily follows.

COROLLARY 13.4. If χ is real and odd, then $S_{21}(\chi, 1) > 0$, if $\chi(2) \neq 1$, and $S_{21}(\chi, 1) = 0$, otherwise.

In view of Corollary 3.3 and the elementary fact that $S_{41} = 0$ if $\chi(2) = -1$ [8], at least part of Corollary 13.4 is expected. If p is a prime, Corollary 13.4 shows that the sum of the quadratic residues modulo p exceeds the sum of the non-residues on $(0, p/2)$ if $p \equiv 3 \pmod 8$, while the two sums are equal if $p \equiv 7 \pmod 8$.

THEOREM 13.5. Let χ be odd. Then

$$S_{31}(\chi, 1) = -\frac{iG(\chi)k}{\pi}\left\{\frac{1}{6}[1-\bar{\chi}(3)]\, L(1, \bar{\chi}) + \frac{3^{1/2}}{4\pi} L(2, \bar{\chi}_{3k})\right\}.$$

Proof. In (12.2), put $f(x) = x$, $c = 0$, and $d = k/3$. The result follows from the same type of calculation as above.

COROLLARY 13.6. If χ is real and odd, then $S_{31}(\chi, 1) > 0$.

The following theorems are proved in the same manner as above.

THEOREM 13.7. Let χ be odd. Then

$$S_{32}(\chi, 1) = \frac{iG(\chi)k}{\pi}\left\{\frac{1}{6}[\bar{\chi}(3)-1]\, L(1, \bar{\chi}) + \frac{3^{1/2}}{2\pi} L(2, \bar{\chi}_{3k})\right\}.$$

COROLLARY 13.8. If χ is real and odd and if $\chi(3) = 1$, then $S_{32}(\chi, 1) < 0$.

THEOREM 13.9. Let χ be even. Then

$$S_{42}(\chi, 1) = -\frac{kG(\chi)}{4\pi}\left\{L(1, \bar{\chi}_{4k}) + \frac{1}{\pi}[2-\bar{\chi}(2)]\left[1 - \frac{1}{4}\bar{\chi}(2)\right] L(2, \bar{\chi})\right\}.$$

COROLLARY 13.10. For χ real and even, we have $S_{42}(\chi, 1) < 0$.

THEOREM 13.11. Let χ be odd. Then

$$S_{41}(\chi, 1) = -\frac{iG(\chi)k}{2\pi}\left\{\frac{1}{4}\bar{\chi}(2)[1-\bar{\chi}(2)]\, L(1, \bar{\chi}) + \frac{1}{\pi} L(2, \bar{\chi}_{4k})\right\}$$

and

$$S_{43}(\chi, 1) = \frac{iG(\chi)k}{2\pi}\left\{[\bar{\chi}(2)-1]\left[\frac{3}{4}\bar{\chi}(2) - 1\right] L(1, \bar{\chi}) + \frac{1}{\pi} L(2, \bar{\chi}_{4k})\right\}.$$

COROLLARY 13.12. Let χ be real and odd. If $\chi(2) \neq -1$, then $S_{41}(\chi, 1) > 0$; in any case, $S_{43}(\chi, 1) < 0$.

THEOREM 13.13. Let χ be even. Then

$$S_{11}(\chi, 2) = \frac{G(\chi)k^2}{\pi^2} L(2, \bar{\chi})$$

and

$$S_{21}(\chi, 2) = \frac{G(\chi)k^2}{4\pi^2}\{\bar{\chi}(2) - 2\}\, L(2, \bar{\chi}).$$

COROLLARY 13.14. If χ is even and real, then $S_{11}(\chi, 2) > 0$ and $S_{21}(\chi, 2) < 0$.

THEOREM 13.15. Let χ be odd. Then

$$S_{11}(\chi, 2) = \frac{iG(\chi)k^2}{\pi} L(1, \bar{\chi}) \tag{13.1}$$

and

$$S_{21}(\chi, 2) = \frac{iG(\chi)k^2}{\pi}\left\{\frac{1}{4}[\bar{\chi}(2)-1]L(1, \bar{\chi}) + \frac{1}{\pi^2}\left[1 - \frac{1}{8}\bar{\chi}(2)\right]L(3, \bar{\chi})\right\}.$$

COROLLARY 13.16. Let χ be odd and real. Then in all cases, $S_{11}(\chi, 2) < 0$; if $\chi(2) = 1$, then $S_{21}(\chi, 2) < 0$.

If χ is real, the class number formula corresponding to (13.1) is due to Cauchy [17]. Pepin [51, p. 205], Lerch [44, p. 395], and Ayoub, Chowla, and Walum [3] have also given proofs of (13.1). Of course, any number of formulas could be proven for $\sum_{a \leq n \leq b} \chi(n) n^r$, where r is a positive integer and a and b are rational multiples of k. However we are unable to make any more non trivial deductions about the positivity (or negativity) of such character sums. In this connection, see [3] and [25].

14. Some questions and problems

In the foregoing work, in order to determine if S_{ji} is of constant sign for classes of real, primitive characters, we expressed S_{ji} as a linear combination of L-functions of real characters evaluated as $s = 1$, and then we inspected the coefficients in this linear combination to determine if all were either non-negative or non-positive. In fact, S_{ji} may always be expressed as a linear combination of L-functions evaluated at $s = 1$. However, in the general situation, the L-functions are associated with complex characters. When non-real characters arise in the representation of S_{ji}, we are unable to say anything about the sign of S_{ji}. We have attempted to find all instances when S_{ji} can be expressed in terms of L-functions of real characters. It is natural to ask if these cases are the only instances when theorems about the non-negativity or non-positivity of S_{ji} are possible. Results of P. D. T. A. Elliott (written communication) appear to indicate that this, indeed, is the case. For example, he has proved the following result. Consider the set of primes p in any residue class, e.g., $p \equiv 1 \pmod 8$, and the associated characters χ_p of a given fixed order. Then the values of $\arg L(1, \chi_p)$, as p varies, are everywhere dense modulo 2π.

Let us look at just one example where the admittedly scant, numerical evidence seems to suggest otherwise. Let $\chi(n)$ denote the Legendre symbol modulo p, where $p \equiv 1 \pmod 4$. Then S_{51} cannot be expressed in terms of L-functions with real characters. However, for $p \equiv 1 \pmod 8$ and $p \leqq 30{,}000$, computations show that $S_{51} > 0$. Sufficient conditions for the positivity of S_{51} are that the two series on the right side of (5.14) are positive. For $p \equiv 1 \pmod 8$, are these two series always positive?

There are a few instances for which we are able to express S_{ji} in terms of L-functions of real characters and for which we are unable to deduce any theorems on the sign of S_{ji}, but for which numerical computations suggest a constant sign. Again, let $\chi(n) = \left(\frac{n}{p}\right)$. For primes p with $p \equiv 7 \pmod 8$ and $p \leqq 200{,}000$, calculations of Duncan Buell show that $h(-5p) < \left\{5 - \left(\frac{5}{p}\right)\right\} h(-p)$, or, equivalently, by Corollary 5.3, that $S_{51} > 0$. Is this true for all p with $p \equiv 7 \pmod 8$?

There are 7 additional cases for intervals of length $p/24$ in which numerical calculations for $p \leqq 30{,}000$ suggest that $S_{24,i}$ may possibly have a constant sign. For $p \equiv 11 \pmod{24}$, $S_{24,3}$, $S_{24,11} > 0$; for $p \equiv 17 \pmod{24}$, $S_{24,8}$, $S_{24,9} < 0$; for $p \equiv 19 \pmod{24}$, $S_{24,6} > 0$; and for $p \equiv 23 \pmod{24}$, $S_{24,2} = -S_{24,12} > 0$. It can be shown that the above inequalities have the following implications, which we very tenuously conjecture hold for all primes in the given residue classes. If $p \equiv 11 \pmod{12}$, then $h(-12p) < 2h(-8p) + h(-24p)$; if $p \equiv 11 \pmod{24}$, then $h(-8p) < 2h(-p) + h(-12p)$; if $p \equiv 17 \pmod{24}$, then $2h(-3p) < 2h(-8p) + h(-24p)$ and $h(-8p) < 2h(-3p) + h(-4p)$; and if $p \equiv 19 \pmod{24}$, then $4h(-p) < h(-12p) + h(-24p)$.

S. Chowla has conjectured that if p is a prime with $p \equiv 3 \pmod 8$, then S_{21} assumes every value that is a positive, odd multiple of 3. He has also conjectured that if $p \equiv 7 \pmod 8$, then S_{21} assumes every positive, odd integral value. In other words, Chowla has conjectured that $h(-p)$ assumes every possible odd value for each of the sets of primes p with $p \equiv 3 \pmod 8$ and $p \equiv 7 \pmod 8$. Samuel Wagstaff has done some calculations to test Chowla's conjectures and similar conjectures of the author. All of the calculational data are for $p \leqq 30{,}000$. For $p \equiv 3 \pmod 8$, the largest value for S_{21} is 297. There are only two omissions, 249 and 291.

For $p \equiv 7 \pmod 8$, the largest value for S_{21} is 259. The smallest value not assumed is 163. There are several other values between 163 and 259 that are not assumed. The calculations also strongly support the following conjectures. S_{41} and S_{31}, for $p \equiv 1 \pmod 4$; S_{52}, for $p \equiv 3 \pmod 4$; S_{81}, for $p \equiv 1 \pmod 8$; S_{82}, for $p \equiv 7 \pmod 8$; $-S_{84}$, for $p \equiv 5 \pmod 8$; and $S_{12,2}$, for $p \equiv 7 \pmod 8$ and for $p \equiv 11 \pmod{12}$, each assumes all positive, integral values. We refer the reader to the foregoing work here for the translations of these conjectures into conjectures about class numbers.

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