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divisor of exponential polynomials  $f$  and  $g$ , then the set of zeros of  $h$  is all but at most finitely many of the common zeros of  $f$  and  $g$ . We have shown this to be the case if at least one of  $f$  and  $g$  is a simple exponential sum.

We see that a natural formulation of the Shapiro problem is: *If  $f$  and  $g$  are exponential polynomials, is it the case that there exists an exponential polynomial  $h$ , the set of zeros of which is exactly the set of common zeros of  $f$  and  $g$ ?*

We recall that it is not, without qualification, the case that if every zero of  $f \in E'$  is a zero of  $g \in E'$  then  $f$  divides  $g$  in the ring  $E'$ ; for example  $(1 - e^z)/z$  is not an element of  $E'$  (its set of integer zeros is not a finite union of arithmetic progressions). Equivalently, it follows that if  $\prod_{l=1}^m (e^{z/2^l} + 1)$  divides an exponential polynomial  $g(z)$  in the ring  $E'$  for all  $m = 1, 2, \dots$  then  $1 - e^z$  divides  $g(z)$  in  $E'$ .

The ideas we have mentioned attack an apparently analytic problem by essentially algebraic methods. Indeed, in a sense, "approximate" methods appear doomed to failure by virtue of the following proposition mentioned to the authors by H. L. Montgomery:

**PROPOSITION 3.** *Let  $\mu(r)$  be any positive-real-valued function decreasing to 0 as  $r \rightarrow \infty$ . Then there exist exponential polynomials  $f, g$  such that for every  $r_0 > 0$  there is an  $r > r_0$  and a  $z \in \mathbb{C}$  with  $r_0 < |z| < r$  such that  $0 < |f(z) - g(z)| \leq \mu(r)$ .*

*Proof.* Define an increasing sequence  $\{n_l\}$  of integers by  $n_0 = 0$  and  $n_{l+1} - n_l \geq -\log(\mu(2^{n_l})/2\pi)/\log 2$  and write  $\alpha = \sum_{l=0}^{\infty} (-1)^l 2^{-n_l}$ . Let  $f(z) = 1 - e^{2\pi i z}$  and  $g(z) = 1 - e^{2\pi i \alpha z}$ , and write  $z_l = 2^{n_l}$ ,  $l = 0, 1, 2, \dots$ . Then  $f(z_l) = 0$  and  $0 < |g(z_l)| = |1 - e^{2\pi i \alpha z_l}| = 2 |\sin \pi \alpha z_l| \leq \mu(2^{n_l})$ , as required. One notices that  $f(z), g(z)$  have the property that there are infinitely many pairs  $z_l, z'_l$  with  $f(z_l) = 0$ ,  $g(z'_l) = 0$  and  $|z_l - z'_l| \leq 2\mu(|z_l|)$ .

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