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divisor of exponential polynomials f and g, then the set of zeros of h is all but at most finitely many of the common zeros of f and g. We have shown this to be the case if at least one of f and g is a simple exponential sum.

We see that a natural formulation of the Shapiro problem is: *If f and g are exponential polynomials, is it the case that there exists an exponential polynomial h, the set of zeros of which is exactly the set of common zeros of f and g?*

We recall that it is not, without qualification, the case that if every zero of $f \in E'$ is a zero of $g \in E'$ then f divides g in the ring E'; for example $(1-e^z)/z$ is not an element of E' (its set of integer zeros in not a finite union of arithmetic progressions). Equivalently, it follows that if $\Pi_{l=1}^{m} (e^{z/2^l}+1)$ divides an exponential polynomial $g(z)$ in the ring E' for all $m = 1, 2, \ldots$ then $1 - e^z$ divides $g(z)$ in E'.

The ideas we have mentioned attack an apparently analytic problem by essentially algebraic methods. Indeed, in a sense, "approximate" methods appear doomed to failure by virtue of the following proposition mentioned to the authors by H. L. Montgomery:

PROPOSITION 3. *Let $\mu(\mathrm{r})$ be any positive-real-valued function decreasing to 0 as $\mathrm{r} \to \infty$. Then there exist exponential polynomials f, g such that for every $\mathrm{r}_0 > 0$ there is an $\mathrm{r} > \mathrm{r}_0$ and a $z \in \mathbf{C}$ with $\mathrm{r}_0 < |z| < \mathrm{r}$ such that $0 < |f(z) - g(z)| \leqslant \mu(\mathrm{r})$.*

Proof. Define an increasing sequence $\{n_l\}$ of integers by $n_0 = 0$ and $n_{l+1} - n_l \geqslant -\log(\mu(2^{n_l})/2\pi)/\log 2$ and write $\alpha = \sum_{l=0}^{\infty} (-1)^l 2^{-n_l}$. Let $f(z) = 1 - e^{2\pi i z}$ and $g(z) = 1 - e^{2\pi i \alpha z}$, and write $z_l = 2^{n_l}$, $l = 0, 1, 2, \ldots$. Then $f(z_l) = 0$ and $0 < |g(z_l)| = |1 - e^{2\pi i \alpha z_l}| = 2|\sin \pi \alpha z_l| \leqslant \mu(2^{n_l})$, as required. One notices that $f(z), g(z)$ have the property that there are infinitely many pairs z_l, z'_l with $f(z_l) = 0$, $g(z'_l) = 0$ and $|z_l - z'_l| \leqslant 2\mu(|z_l|)$.

REFERENCES

[1] BERENSTEIN, C. A. and M. A. DOSTAL. A lower Estimate for Exponential Sums. *Bull. Amer. Math. Soc. 80* (1974), pp. 687-691.

[2] DICKSON, D. G. Asymptotic Distribution of Exponential Sums. *Publ. Math. Debrecen 11* (1964), pp. 295-300.

[3] JAGER, H. A Note on the Vanishing of Power Sums. *Ann. Univ. Sci. Bud sect. Math. 10* (1967), pp. 13-16.

[4] LECH, C. A Note on Recurring Series. *Ark. Mat. 2* (1953), pp. 417-421.

[5] MAHLER, K. Eine Arithmetische Eigenschaft der Taylorkoeffizienten rationaler Funktionen. *Proc. Ac. Amsterdam 38* (1935), pp. 51-60.

[6] —— On the Taylor coefficients of rational functions. *Proc. Camb. Phil. Soc. 52* (1956), pp. 39-48.

[7] MORENO, C. J. The Zeros of Exponential Polynomials. *Comp. Math. 26* (1973), pp. 69-78.

[8] POLYA, G. Geometrisches über die Verteilung der Nullstellen gewisser ganzer transzendenter Funktionen. *Münchener Sitzungsberichte 50* (1920), pp. 285-290.

[9] van der POORTEN, A. J. A Note on the Zeros of Exponential Polynomials. (to appear).

[10] RITT, J. F. A Factorisation theory for Functions $\Sigma_{i=1}^{n} a_i e^{\alpha_i z}$. *Trans. Amer. Math. Soc. 29* (1927), pp. 584-596.

[11] —— On the Zeros of Exponential Polynomials. *Trans. Amer. Math. Soc. 31* (1929), pp. 680-686.

[12] —— Algebraic Combinations of Exponentials. *Trans. Amer. Math. Soc. 31* (1929), pp. 654-679.

[13] SHAPIRO, H. S. The Expansion of mean-periodic functions in series of exponentials. *Comm. Pure and Appl. Math. 11* (1958), pp. 1-21.

[14] SHIELDS, A. On Quotients of Exponential Polynomials. *Comm. Pure and Appl. Math. 16* (1963), pp. 27-31.

[15] SKOLEM, Th. Ein Verfahren zur Behandlung gewisser Exponentialer Gleichungen und Diophantischer Gleichungen. *C.r. 8 congr. scand. à Stockholm. 1934*, pp. 163-188.

[16] TIJDEMAN, R. On a Conjecture of Turán and Erdös. *Indag. Math. 28* (1966), pp. 374-383.

[17] —— On the Number of Zeros of General Exponential Polynomials. *Indag. Math. 33* (1971), pp. 1-7.

[18] TITCHMARSH, E. C. *The Theory of Functions.* Oxford Univ. Press, 1952.

[19] SHAPIRO, H. N. On a Theorem concerning Exponential Polynomials. *Comm. Pure and Appl. Math. 12* (1959), pp. 487-500.

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