1. Introduction

ON COMMON ZEROS OF EXPONENTIAL POLYNOMIALS

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1. INTRODUCTION

At the 1974 Bolyai Janos Society Colloquium on Number Theory, H. L. Montgomery mentioned the following problem, which he attributed to H. S. Shapiro [13]:

Denote by E the collection of all exponential polynomials

$$(1)\quad E = \{a_1 e^{\alpha_1 z} + a_2 e^{\alpha_2 z} + \ldots + a_n e^{\alpha_n z} : a_1, \ldots, a_n \in \mathbf{C}, \alpha_1, .., \alpha_n \in \mathbf{C}, n \in \mathbf{N}\}$$

Suppose $f, g \in E$ have infinitely many zeros in common. Then is it the case that there exists an h in E, such that h has infinitely many zeros, and h is a common factor of f and g in the ring E?

As we see below, it is equivalent to ask whether there exists an h in E, such that h has infinitely many zeros, and such that all the zeros of h are common zeros of f and g. Henceforth in this note we refer to Shapiro's problem simply as "the problem".

The problem is mentioned by H. S. Shapiro [13] in the context of his study of mean-periodic functions satisfying a certain functional equation. There, [13], p. 18 the problem appears in the form of a conjecture:

If two exponential polynomials have infinitely many zeros in common they are both multiples of some third (entire transcendental) exponential polynomial.

In this note we survey those ideas that appear relevant to settling this conjecture. Many of the ideas we mention here independently in response to Montgomery's question, are already alluded to in [13]. In particular we should remark that the conjecture arises as a generalisation of the Skolem-Mahler-Lech theorem which we describe in Section 4.

In the sequel we refer to the quantities $\alpha_1, \ldots, \alpha_n$ in (1) as the *frequencies* of the exponential polynomial, and the quantities $a_1, \ldots, a_n$ as the *coefficients*. Unless otherwise indicated we shall always suppose given frequencies to be distinct and given coefficients to be non-zero. Similarly we shall suppose an exponential polynomial to have at least two distinct terms, hence to have a zero, and indeed hence to have infinitely many zeros. We mention some results on zeros of exponential polynomials in section 2.

As Professor Turán pointed out to us, one source of information on the problem is the papers of J. F. Ritt. Ritt provides a factorisation theory for exponential polynomials [10], and shows *inter alia* that if a quotient of exponential polynomials is an entire function then it is an exponential polynomial [11]. We describe these results in section 3.

In the case where all the frequencies of one of the exponential polynomials are rational we confirm that there is indeed a common factor. Even this special case seems to require a non-trivial argument; we employ the theorem of Skolem-Mahler-Lech on recurrence sequences with infinitely many vanishing terms (Lech [4], Mahler [6]). Conversely we observe in section 4 that an affirmative answer to the problem implies a generalised form of the Skolem-Mahler-Lech theorem. For a similar application of this theorem to zeros of exponential polynomials see Jager [3].

It follows from the results mentioned in sections 3 and 4 that one can define the greatest common divisor $h \in E$ of two exponential polynomials $f, g \in E$. An affirmative answer to the problem then implies that the set of zeros of the gcd h is all but at most finitely many of the common zeros of f and g. We make these and other remarks in section 5. We conclude this section with an example due to Montgomery which shows that "approximate methods" in the obvious manner are doomed to failure.

2. Zeros of Exponential Polynomials

Given an exponential polynomial,

$$f(z) = \Sigma a_j e^{\alpha_j z} = a_1 e^{\alpha_1 z} + \dots + a_n e^{\alpha_n z}$$

denote by C_f the convex polygon in the complex plane defined by the complex conjugates of the frequencies; that is, the convex hull of the points $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n$. Then *the zeros of f lie in half-strips in the directions of the exterior normals to C_f*. More quantitatively, *suppose an edge of the polygon C_f has length l. Then the number of zeros of $f(z)$ in the half-strip perpendicular to that edge and of absolute value less than R is*

$$(2) \qquad \frac{lR}{2\pi} + O(1) \qquad ; \text{ see Pólya [8], D. G. Dickson [2].}$$

It can also be shown that near every line in and parallel to the sides of a strip of zeros lie infinitely many zeros of the exponential polynomial, see Moreno [7], van der Poorten [9]. From a different point of view, one can