ON COMMON ZEROS OF EXPONENTIAL POLYNOMIALS

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1. INTRODUCTION

At the 1974 Bolyai Janos Society Colloquium on Number Theory, H. L. Montgomery mentioned the following problem, which he attributed to H. S. Shapiro [13]:

Denote by E the collection of all exponential polynomials

(1) $E = \{a_1 e^{\alpha_1 z} + a_2 e^{\alpha_2 z} + \ldots + a_n e^{\alpha_n z} : a_1, \ldots, a_n \in \mathbf{C}, \alpha_1, \ldots, \alpha_n \in \mathbf{C}, n \in \mathbf{N}\}$

Suppose $f, g \in E$ have infinitely many zeros in common. Then is it the case that there exists an h in E, such that h has infinitely many zeros, and h is a common factor of f and g in the ring E?

As we see below, it is equivalent to ask whether there exists an h in E, such that h has infinitely many zeros, and such that all the zeros of h are common zeros of f and g. Henceforth in this note we refer to Shapiro's problem simply as "the problem".

The problem is mentioned by H. S. Shapiro [13] in the context of his study of mean-periodic functions satisfying a certain functional equation. There, [13], p. 18 the problem appears in the form of a conjecture:

If two exponential polynomials have infinitely many zeros in common they are both multiples of some third (entire transcendental) exponential polynomial.

In this note we survey those ideas that appear relevant to settling this conjecture. Many of the ideas we mention here independently in response to Montgomery's question, are already alluded to in [13]. In particular we should remark that the conjecture arises as a generalisation of the Skolem-Mahler-Lech theorem which we describe in Section 4.

In the sequel we refer to the quantities $\alpha_1, \ldots, \alpha_n$ in (1) as the *frequencies* of the exponential polynomial, and the quantities $a_1, \ldots, a_n$ as the *coefficients*. Unless otherwise indicated we shall always suppose given frequencies to be distinct and given coefficients to be non-zero. Similarly we shall suppose an exponential polynomial to have at least two distinct terms, hence to have a zero, and indeed hence to have infinitely many zeros. We mention some results on zeros of exponential polynomials in section 2.

As Professor Turán pointed out to us, one source of information on the problem is the papers of J. F. Ritt. Ritt provides a factorisation theory for exponential polynomials [10], and shows *inter alia* that if a quotient of exponential polynomials is an entire function then it is an exponential polynomial [11]. We describe these results in section 3.

In the case where all the frequencies of one of the exponential polynomials are rational we confirm that there is indeed a common factor. Even this special case seems to require a non-trivial argument; we employ the theorem of Skolem-Mahler-Lech on recurrence sequences with infinitely many vanishing terms (Lech [4], Mahler [6]). Conversely we observe in section 4 that an affirmative answer to the problem implies a generalised form of the Skolem-Mahler-Lech theorem. For a similar application of this theorem to zeros of exponential polynomials see Jager [3].

It follows from the results mentioned in sections 3 and 4 that one can define the greatest common divisor $h \in E$ of two exponential polynomials $f, g \in E$. An affirmative answer to the problem then implies that the set of zeros of the gcd h is all but at most finitely many of the common zeros of f and g. We make these and other remarks in section 5. We conclude this section with an example due to Montgomery which shows that "approximate methods" in the obvious manner are doomed to failure.

2. Zeros of Exponential Polynomials

Given an exponential polynomial,

$$f(z) = \Sigma a_j e^{\alpha_j z} = a_1 e^{\alpha_1 z} + \dots + a_n e^{\alpha_n z}$$

denote by C_f the convex polygon in the complex plane defined by the complex conjugates of the frequencies; that is, the convex hull of the points $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n$. Then *the zeros of f lie in half-strips in the directions of the exterior normals to C_f*. More quantitatively, *suppose an edge of the polygon C_f has length l. Then the number of zeros of $f(z)$ in the half-strip perpendicular to that edge and of absolute value less than R is*

(2) $$\frac{lR}{2\pi} + O(1)$$; see Pólya [8], D. G. Dickson [2].

It can also be shown that near every line in and parallel to the sides of a strip of zeros lie infinitely many zeros of the exponential polynomial, see Moreno [7], van der Poorten [9]. From a different point of view, one can obtain upper bounds for the number of zeros of $f(z)$ in any circle of radius R in the complex plane; if $\Delta = \max_k |\alpha_k|$, then such a bound is

$$3(n-1) + 4R\Delta \qquad ; \text{ see Tijdeman [17].}$$

For further details the reader is referred to the cited articles and the appropriate references mentioned therein.

3. Factorisation of Exponential Polynomials

We describe the results of J. F. Ritt [10], [11].

Define an ordering on the set of complex numbers by: $\alpha < \beta$ if $\mathcal{R}e\,\alpha \leqslant \mathcal{R}e\,\beta$, and if $\mathcal{R}e\,\alpha = \mathcal{R}e\,\beta$ then $\mathcal{I}m\,\alpha < \mathcal{I}m\,\beta$. We will suppose in the sequel, that, unless indicated otherwise, any exponential polynomial

$$\Sigma\, a_j\, e^{\alpha_j z} = a_1\, e^{\alpha_1 z} + \ldots + a_n\, e^{\alpha_n z}$$

is so normalised that $a_1 = 1$ and $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_n$. Of course the normalisation is effected by multiplying by some unit $be^{\beta z}$, $b \neq 0$ thus not affecting the zeros of the exponential polynomial. Many of the remarks below are invalid if the normalisation is not assumed.

Ritt [10] firstly shows that *if the exponential polynomial* $\sum b_j e^{\beta_j z}$ *divides the exponential polynomial* $\sum a_j e^{\alpha_j z}$ *(in the ring E of exponential polynomials) then the frequencies* $\beta_1, \ldots, \beta_m$ *are linear combinations with rational coefficients of the frequencies* $\alpha_1, \ldots, \alpha_n$. Now, following Ritt, call an exponential polynomial $\sum a_j e^{\alpha_j z}$ *simple* if its frequencies are commensurable, that is, there is a minimal (in the sense of the ordering on **C**) number α such that each α_j is a non-negative integer multiple of α. So such a simple exponential polynomial $f(z)$ is a polynomial in $e^{\alpha z}$ and factorises into a finite product of functions of the shape $1 + ae^{\alpha z}$, $a \in \mathbf{C}$. Of course, $1 + ae^{\alpha z}$ has factors of the shape $1 + a' e^{(\alpha/m)z}$ for each $m = 1, 2, 3, \ldots$ but it follows from Ritt's lemma mentioned above that every factor of $f(z)$ is a product of such factors. Similarly, call an exponential polynomial *irreducible* if it has no non-trivial (that is, other than units and associates) factors in the ring E. Then Ritt's principal result is that *an exponential polynomial can be factorised uniquely as a finite product of simple exponential polynomials such that their sets of frequencies are pairwise incommensurable, and a finite product of irreducible exponential polynomials.*

We outline the structure of the proof. Firstly one shows that there exist complex numbers $\mu_1, \mu_2, \ldots, \mu_p$ linearly independent over the field of rational numbers $\mathbf{Q}$ such that each frequency of f is a linear combination of the $\mu_1, ..., \mu_p$ with non-negative integer coefficients. It follows that any normalised factor (in the ring E) of f similarly has frequencies which are linear combinations of the $\mu_1, ..., \mu_p$ with non-negative rational coefficients. Now write $y_1 = e^{\mu_1 z}, ..., y_p = e^{\mu_p z}$. Then $f(z)$ becomes a polynomial $q(y_1, ..., y_p) = q(y)$. Moreover, it is clear that for each finite factorisation of $f(z)$ in the ring E there is, for some set of positive integers $t_1, ..., t_p$ a factorisation of $q(y_1^{t_1}, ..., y_p^{t_p}) = q(y^t)$ in the ring $\mathbf{C}[y_1, ..., y_p] = \mathbf{C}[y]$; conversely to each such factorisation in $\mathbf{C}[y]$ into polynomials with constant term 1, there is a factorisation in E. We suppose henceforth that polynomials have constant term 1. To make the correspondence, observed above, one-one, Ritt defines $q(y) \in \mathbf{C}[y]$ to be *primary* if for each $i = 1, 2, ..., p$ the exponents of y_i in the monomials comprising $q(y)$ have greatest common divisor 1. One sees that if $q(y)$ is primary then of the irreducible factors of $q(y^t)$ in $\mathbf{C}[y]$ either all or none are again primary. Then if $f(z)$ is represented by a primary polynomial $q(y)$ each finite factorisation of $f(z)$ in E corresponds one-one to a minimal choice of $t = (t_1, ..., t_p)$ and a factorisation of $q(y^t)$. Ritt now shows that if $q(y)$ is primary and has more than two terms (including constant term 1) then there are only finitely many sets $t = (t_1, ..., t_p)$ of positive integers such that the irreducible factors of $q(y^t)$ are primary. This settles the finiteness of the factorisation and the remainder of the proof is straightforward.

In [11] Ritt proves that *if a quotient of exponential polynomials is an entire function then it is an exponential polynomial*; in [12] it is shown *inter alia* that it is sufficient that the quotient be regular in a sector of opening greater than π. We remark on generalisations of Ritt's result in section 5. An equivalent assertion to Ritt's theorem is *if every zero of $f(z) = \sum a_j e^{\alpha_j z}$ is a zero of $g(z) = \sum b_j e^{\beta_j z}$ then $f(z)$ divides $g(z)$ in the ring E.* The principle of the proof is as follows: denote by $|C_f|$ the maximal real cross-section, that is, parallel to the real axis, of the polygon C_f defined in section 2. Then one shows there exist exponential polynomials q and r such that $g = qf + r$ and $|C_r| < |C_f|$. It follows that r has less zeros than does f in sufficiently large rectangles, whence $r \equiv 0$ as required.

We should remark that by a different method Allen Shields [14] has shown that a quotient of exponential polynomials is an exponential polynomial already provided that the number of poles of the quotient in $|z| < R$ is $o(R)$. This result follows from the proof outlined above. We further note that H. N. Shapiro [19, §5] has given a division theorem related to Ritt's theorem.

The factorisation theory implies that we need consider only two cases of the Shapiro problem. Namely, firstly the case where at least one of the exponential polynomials f, g is simple. We settle this case, in the affirmative in section 4. Secondly one must take the case where at least one of the exponential polynomials is irreducible. Then an affirmative answer to the problem is equivalent to the truth of the following conjecture:

Let f, g be exponential polynomials and let f be irreducible. Then if f and g have infinitely many zeros in common, f divides g in the ring E (equivalently, g/f is an entire function).

Equivalent to this conjecture is: *if f, g are distinct irreducible exponential polynomials then f and g have at most finitely many common zeros.* This last formulation can be rephrased in terms of polynomials (with constant term 1):

Let $f(y) = f(y_1, \dots, y_p)$, $g(y)$ be distinct polynomials irreducible in $\mathbf{C}[y]$ in the strong sense that for all sets $t_1, \dots, t_p$ of positive integers, the polynomials $f(y^t) = f(y_1^{t_1}, \dots, y_p^{t_p})$, $g(y^t)$ are so irreducible. Denote by $V \subset \mathbf{C}^p$ the set of common zeros of $f(y)$ and $g(y)$. Let $\mu_1, \dots, \mu_p$ be numbers linearly independent over $\mathbf{Q}$. Then the curve $\{(e^{\mu_1 z}, \dots, e^{\mu_p z}) : z \in \mathbf{C}\}$ meets V in at most finitely many points.

4. The Theorem of Skolem-Mahler-Lech

The following result was proved by Skolem [15] for the field of rational numbers, by Mahler [5] for the field of algebraic numbers, and by Lech [4] and Mahler [6] for arbitrary fields of characteristic zero (the assertion is false in fields of characteristic p):

Let $\{c_\nu\}$ be a sequence whose elements lie in a field of characteristic zero and satisfy a linear homogeneous recurrence relation

$$c_\nu = b_1 c_{\nu-1} + b_2 c_{\nu-2} + \dots + b_n c_{\nu-n}, \qquad \nu = n, n+1, \dots \tag{3}$$

Denote by $\mathrm{M} \subset \mathrm{N}$ *the set of indices ν such that* $c_\nu = 0$. *Then* M *is a finite union of arithmetic progressions* (the progressions may have common difference 0 and so consist of a single point). Hence those c_ν equal to zero occur periodically in the sequence from a certain index on.

It is well-known that there exist elements $\beta_1, \dots, \beta_m$, namely the distinct zeros of the polynomial

$$z^n - b_1 z^{n-1} - b_2 z^{n-2} - \dots - b_n, \tag{4}$$

and polynomials $p_1, \dots, p_m$ where $1 + \deg p_j$ is the multiplicity of β_j as a zero of (4), $j = 1, 2, \dots, m$, such that for all $\nu = 0, 1, 2, \dots$

$$c_\nu = \sum_{j=1}^{m} p_j(\nu)\beta_j^{\,\nu} \tag{5}$$

Conversely any sequence $\{c_\nu\}$ where c_ν is given by (5) satisfies a relation of the shape (3). Hence, after writing $\beta_j = e^{\alpha j}$ a special case (namely, where the zeros of (4) are distinct) of the Skolem-Mahler-Lech theorem is:

Lemma. *Let $f(z) = \sum a_j e^{\alpha_j z}$ be an exponential polynomial. Denote by* M $\subset$ **Z** *the subset of the integers* **Z** *on which f vanishes. Then* M *is a finite union of arithmetic progressions* $\{d_0 + nd : n \in \mathbf{Z}\}$; (and if M is infinite at least one common difference d is non-zero).

Proof. The assertion seems broader than the Skolem-Mahler-Lech theorem in that we claim that if $f(z)$ vanishes for all $z = d_0 + nd$, $n \in \mathbf{N}$ then it vanishes for all $z = d_0 + nd$, $n \in \mathbf{Z}$. This is not difficult to show directly, see for example [16], theorem 2. However the broader assertion is already implied by the proof of the Skolem-Mahler-Lech theorem which we outline for its intrinsic interest: Let K be the field $K = \mathbf{Q}(e^{\alpha_1}, \dots, e^{\alpha_n}; a_1 \dots, a_n)$; then one shows there is a valuation $|\ |_{\mathfrak{p}}$ of K such that $|p|_{\mathfrak{p}} = 1/p$ for some rational prime p and

$$|e^{\alpha_j d} - 1|_{\mathfrak{p}} < p^{-1/(p-1)}, \qquad j = 1, 2, \dots, n,$$

for some natural number $d > 0$. Now consider the $\mathfrak{p}$-adic functions $f_l : z \mapsto \sum a_j \exp(\alpha_j l) \exp(d\alpha_j z)$, $l = 0, 1, \dots d-1$. These functions are well-defined by $\mathfrak{p}$-adic power series converging for $z \in \mathbf{Z}_p$, the p-adic integers. If $f(z)$ vanishes at infinitely many integers then some f_l, say f_{d_0}, has infinitely many zeros in the compact set $\mathbf{Z}_p$. Hence the $\mathfrak{p}$-adic power series f_{d_0} vanishes identically, so $f(z)$ vanishes on the set $\{d_0 + nd : n \in \mathbf{Z}\}$ as asserted.

It follows from results mentioned in section 3 that if $f(z)$ vanishes for $z = d_0 + nd$, $n \in \mathbf{Z}$ then the exponential polynomial

$$1 - \exp(2\pi i/d)\, d_0 \exp(2\pi i/d)\, z$$

divides $f(z)$ in the ring E.

Theorem. *Let $f(z) = \sum a_j e^{\alpha_j z}$ be an exponential polynomial with pairwise commensurable frequencies (a simple exponential polynomial) and let $g(z)$ be an arbitrary exponential polynomial such that $f(z)$ and $g(z)$ have infinitely many common zeros. Then there exists an exponential polynomial $h(z)$, with infinitely many zeros, such that h is a common factor of f and g in the ring of exponential polynomials, E.*

Information on the frequencies of h can be deduced from [19].

Proof. The commensurability of the frequencies implies there is a number α such that all the frequencies $\alpha_1, \ldots, \alpha_n$ are positive integer multiples of α. Then $f(z)$ is a polynomial in $e^{\alpha z}$ and can be factorised as a finite product of factors of the shape $1 - ae^{\alpha z}$. Since $f(z)$, $g(z)$ have infinitely many common zeros, at least one of these factors, say $1 - ae^{\alpha z}$, has infinitely many zeros in common with $g(z)$. So $g(z)$ has infinitely many zeros of the shape $z = (2k\pi i - \log a)/\alpha$, $k \in \mathbf{Z}$. Hence the exponential polynomial $g^*(z) = g\big((2\pi iz - \log a)/\alpha\big)$ vanishes on an infinite subset M of $\mathbf{Z}$, and by the lemma it follows that $g^*(z)$ vanishes on an arithmetic progression $\{d_0 + nd : n \in \mathbf{Z}\}$, $d \neq 0$. Then, as remarked above, the exponential polynomial $h^*(z) = 1 - \exp(2\pi i/d)\, d_0 \exp(2\pi i/d)\, z$ divides $g^*(z)$ in the ring E. It follows that the exponential polynomial $h(z) = 1 - \exp\big((2\pi i/d)\, d_0 + (1/d) \log a\big) . e^{(\alpha/d)z}$ divides $g(z)$ in E. Since $h(z)$ divides $1 - ae^{\alpha z}$, and *a fortiori* $f(z)$, we have the assertion.

We shall show in section 5 that, conversely, the theorem implies the Skolem-Mahler-Lech theorem for sequences $\{c_\nu\}$ where $c_\nu = \sum b_j e^{\beta_j z}$, the coefficients b_j being constants. This observation leads us to remark that, more generally an affirmative answer to the problem implies the following:

Suppose that the exponential polynomials f, g have infinitely many zeros in common. Then the common zeros are located in a finite number of half-strips. Further for each such half-strip the common zeros are distributed "almost periodically" in the sense that there is a constant c such that the number of common zeros in the half-strip which are in absolute value less than R is $cR + O(1)$.

This remark, which follows immediately from (2) in section 2 can be considered as a generalisation of the Skolem-Mahler-Lech theorem. Since, in general, we do not know sufficient conditions for some infinite set of points to be the zeros of an exponential polynomial this generalisation tells only part of the conjectured truth.

5. Further Remarks

In this note we have considered the ring E, often called the ring of exponential sums, though it is arguably more natural to consider the ring

$$E' = \{a_1(z)\, e^{\alpha_1 z} + \ldots + a_n(z)\, e^{\alpha_n z} : a_1(z), \ldots, a_n(z) \in \mathbf{C}[z], \alpha_1, \ldots, \alpha_n \in \mathbf{C}, n \in \mathbf{N}\},$$

more properly called the ring of exponential polynomials. Indeed E' has the very natural description: $f \in E'$ if and only if f satisfies a homogeneous linear differential equation with constant coefficients. The results mentioned in section 2 generalise *mutatis mutandis* to apply to the ring E'. Similarly, the factorisation theory of section 3 generalises to apply to the ring E'; one need only observe that if $\sum a_j(z)e^{\alpha_j z}$ factorises non-trivially in E' then $\sum a_j(\beta)e^{\alpha_j z}$ must factorise in E for all $\beta \in \mathbf{C}$; or one applies Ritt's argument in the polynomial ring $\mathbf{C}[z][y_1, ..., y_p]$ rather than $\mathbf{C}[y_1, ..., y_p]$. Furthermore, it is known that if g/f is an entire function, where $g, f \in E'$ then $g/f = h/a$ where $h \in E'$ and, if $f(z) = \sum a_j(z)e^{\alpha_j z}$, then a is a polynomial such that a divides gcd $(a_1(z), ..., a_n(z))$; indeed this result is valid in the ring of general exponential polynomials in several complex variables, see Berenstein and Dostal [1] for details and references. Finally, we note that the Skolem-Mahler-Lech theorem applies to elements of E' so that the theorem of section 4 generalises to state that if a simple exponential sum (necessarily in E) and any general exponential polynomial (in E') have infinitely many common zeros than they have a common divisor (which, by the proof, lies in E). Below we refer to elements of E' as exponential polynomials and refer to elements of the subring E as exponential sums.

PROPOSITION 1. *The assertion that, if a simple exponential sum and an exponential polynomial have infinitely many zeros in common then they have a non-trivial common divisor in the ring E', is equivalent to the Skolem-Mahler-Lech theorem.*

Proof. In one direction the implication is the content of the theorem of section 4 and the remarks above. Conversely, take, without loss of generality, the exponential sum to be $1 - e^z$ and consider the exponential polynomial as the product of its Ritt factors, that is, a polynomial, a finite number of simple exponential sums whose sets of frequencies are pairwise incommensurable, and a finite number of irreducible exponential polynomials. Firstly, $1 - e^z$ and an irreducible exponential polynomial can have at most finitely many common zeros because otherwise the irreducible exponential polynomial has a non-trivial divisor in E. Secondly, $1 - e^z$ and a polynomial, obviously have at most finitely many common zeros. Thirdly, a simple exponential sum is a finite product of terms of the shape $1 - ae^{\alpha z}$; if α is irrational so that 1 and α are incommensurable, then $1 - e^z$ and $1 - ae^{\alpha z}$ have at most one common zero. On the other hand, if α is rational, say $\alpha = r/d$, then the common zeros of $1 - e^z$ and $1 - ce^{\alpha z}$ are the zeros of finitely many functions of the shape $1 - \exp(2\pi i d_0/d) \exp z/d$ and so occur in arithmetic progressions. Hence the common zeros are a finite union of arithmetic progressions (which may have common difference zero). In particular, if an exponential polynomial has infinitely many integer zeros, and so, infinitely many zeros in common with $1 - e^{2\pi i z}$ then these integer zeros are a finite union of arithmetic progressions, and this is the content of the Skolem-Mahler-Lech theorem.

Proposition 2. *Every pair f, g of exponential polynomials has a greatest common divisor* (gcd) *h in the ring E'* (in the usual sense that h is a common divisor of f and g in E' and every common divisor of f and g in E' divides h in E').

Proof. The Ritt factorisation theory implies one need on y consider the cases where f is a polynomial, a simple exponential sum, or an irreducible exponential polynomial. If f is a polynomial the gcd is again a polynomial, and if f is irreducible it is a unit or an associate of f. Finally if f is simple then the gcd is a product of a polynomial and a finite number of functions of the shape of $h(z)$ as constructed in the proof of the theorem of section 4, that is, of functions the set of zeros of each of which 's an arithmetic progression.

Shields [14] remarks that the above proposition has been obtained by W. D. Bouwsma (unpublished).

We call the abovementioned greatest common divisor the "Ritt gcd" of the two exponential polynomials f and g, and observe that one can also define a function-theoretic gcd of f and g as follows: (see, for example, Titchmarsh [18], Chapter 8).

Let $z_1, z_2, \ldots$ be the common zeros of f and g. Then the exponent of convergence ρ' of these numbers is at most the exponent of convergence of the zeros of f, hence at most the order of f. Thus $\rho' \leqslant 1$. By the Weierstrass factorisation theorem the canonical product h of $z_1, z_2, \ldots$ is an analytic function, and by Borel's theorem the order ρ of h equals ρ'. By virtue of the Hadamard factorisation theorem every entire function of order $\rho \leqslant 1$ with zeros $z_1, z_2, \ldots$ and no others is the product of $h(z)$ and a unit factor of the shape $e^{\alpha + \beta z}$. Hence $h(z)$ is uniquely determined up to a normalisation. We call the function $h(z)$ so defined the "Hadamard gcd" of the functions f and g. The Shapiro problem can now be posed as follows: *Is it the case that apart from a possible polynomial factor, the Hadamard* gcd *of two exponential polynomials coincides with their Ritt* gcd*?* It is equivalent to ask whether the Hadamard gcd of two exponential polynomials is indeed an exponential polynomial and so has exact order 0 or 1.

Our last remark depends on the observation that an affirmative answer to the problem implies: *if the exponential polynomial h is the greatest common* *divisor of exponential polynomials f and g, then the set of zeros of h is all but at most finitely many of the common zeros of f and g.* We have shown this to be the case if at least one of f and g is a simple exponential sum.

We see that a natural formulation of the Shapiro problem is: *If f and g are exponential polynomials, is it the case that there exists an exponential polynomial h, the set of zeros of which is exactly the set of common zeros of f and g?*

We recall that it is not, without qualification, the case that if every zero of $f \in E'$ is a zero of $g \in E'$ then f divides g in the ring E'; for example $(1-e^z)/z$ is not an element of E' (its set of integer zeros in not a finite union of arithmetic progressions). Equivalently, it follows that if $\Pi_{l=1}^{m} (e^{z/2^l}+1)$ divides an exponential polynomial $g(z)$ in the ring E' for all $m = 1, 2, \ldots$ then $1 - e^z$ divides $g(z)$ in E'.

The ideas we have mentioned attack an apparently analytic problem by essentially algebraic methods. Indeed, in a sense, "approximate" methods appear doomed to failure by virtue of the following proposition mentioned to the authors by H. L. Montgomery:

PROPOSITION 3. *Let* $\mu(\mathrm{r})$ *be any positive-real-valued function decreasing to* 0 *as* $\mathrm{r} \to \infty$. *Then there exist exponential polynomials* f, g *such that for every* $\mathrm{r}_0 > 0$ *there is an* $\mathrm{r} > \mathrm{r}_0$ *and a* $z \in \mathbf{C}$ *with* $\mathrm{r}_0 < |z| < \mathrm{r}$ *such that* $0 < |f(z) - g(z)| \leqslant \mu(\mathrm{r})$.

Proof. Define an increasing sequence $\{n_l\}$ of integers by $n_0 = 0$ and $n_{l+1} - n_l \geqslant -\log(\mu(2^{n_l})/2\pi)/\log 2$ and write $\alpha = \sum_{l=0}^{\infty}(-1)^l 2^{-n_l}$. Let $f(z) = 1 - e^{2\pi i z}$ and $g(z) = 1 - e^{2\pi i \alpha z}$, and write $z_l = 2^{n_l}$, $l = 0, 1, 2, \ldots$. Then $f(z_l) = 0$ and $0 < |g(z_l)| = |1 - e^{2\pi i \alpha z_l}| = 2|\sin \pi \alpha z_l| \leqslant \mu(2^{n_l})$, as required. One notices that $f(z), g(z)$ have the property that there are infinitely many pairs z_l, z'_l with $f(z_l) = 0$, $g(z'_l) = 0$ and $|z_l - z'_l| \leqslant 2\mu(|z_l|)$.

REFERENCES

[1] BERENSTEIN, C. A. and M. A. DOSTAL. A lower Estimate for Exponential Sums. *Bull. Amer. Math. Soc. 80* (1974), pp. 687-691.

[2] DICKSON, D. G. Asymptotic Distribution of Exponential Sums. *Publ. Math. Debrecen 11* (1964), pp. 295-300.

[3] JAGER, H. A Note on the Vanishing of Power Sums. *Ann. Univ. Sci. Bud sect. Math. 10* (1967), pp. 13-16.

[4] LECH, C. A Note on Recurring Series. *Ark. Mat. 2* (1953), pp. 417-421.

[5] MAHLER, K. Eine Arithmetische Eigenschaft der Taylorkoeffizienten rationaler Funktionen. *Proc. Ac. Amsterdam 38* (1935), pp. 51-60.

[6] —— On the Taylor coefficients of rational functions. *Proc. Camb. Phil. Soc. 52* (1956), pp. 39-48.

[7] MORENO, C. J. The Zeros of Exponential Polynomials. *Comp. Math. 26* (1973), pp. 69-78.

[8] POLYA, G. Geometrisches über die Verteilung der Nullstellen gewisser ganzer transzendenter Funktionen. *Münchener Sitzungsberichte 50* (1920), pp. 285-290.

[9] van der POORTEN, A. J. A Note on the Zeros of Exponential Polynomials. (to appear).

[10] RITT, J. F. A Factorisation theory for Functions $\Sigma_{i=1}^{n} a_i e^{\alpha_i z}$. *Trans. Amer. Math. Soc. 29* (1927), pp. 584-596.

[11] —— On the Zeros of Exponential Polynomials. *Trans. Amer. Math. Soc. 31* (1929), pp. 680-686.

[12] —— Algebraic Combinations of Exponentials. *Trans. Amer. Math. Soc. 31* (1929), pp. 654-679.

[13] SHAPIRO, H. S. The Expansion of mean-periodic functions in series of exponentials. *Comm. Pure and Appl. Math. 11* (1958), pp. 1-21.

[14] SHIELDS, A. On Quotients of Exponential Polynomials. *Comm. Pure and Appl. Math. 16* (1963), pp. 27-31.

[15] SKOLEM, Th. Ein Verfahren zur Behandlung gewisser Exponentialer Gleichungen und Diophantischer Gleichungen. *C.r. 8 congr. scand. à Stockholm. 1934*, pp. 163-188.

[16] TIJDEMAN, R. On a Conjecture of Turán and Erdös. *Indag. Math. 28* (1966), pp. 374-383.

[17] —— On the Number of Zeros of General Exponential Polynomials. *Indag. Math. 33* (1971), pp. 1-7.

[18] TITCHMARSH, E. C. *The Theory of Functions.* Oxford Univ. Press, 1952.

[19] SHAPIRO, H. N. On a Theorem concerning Exponential Polynomials. *Comm. Pure and Appl. Math. 12* (1959), pp. 487-500.

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