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A direct computation gives the following values

$$\begin{aligned} a &= 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \\ \Delta_a &= 4, -9, -1, -11, -1, 1, 11, 1, 9, -4 \end{aligned}$$

The fifth powers are  $4a = 1, 10$  that is  $a = 3, 8$  and for these  $\Delta_3 = (-3/p)_Z \cdot x = -x = -1$  and  $\Delta_8 = (-8/p)_Z \cdot x = x = 1$  as required.

I should like to thank Professor Frohlich sincerely for his suggestion to look at these  $\Delta_a$ .

## APPENDIX

1. For the convenience of the reader we give here the definition of  $(\alpha/\beta)_{10}$ , the tenth power residue symbol and some of its properties.

First let  $\pi$  be a prime factor of a rational prime  $p \equiv 1 \pmod{5}$ . The residue classes mod  $\pi$ , in  $\mathbf{Z}[\zeta]$ , form a field of norm  $\pi = p$  elements. The non-zero classes form a cyclic group (multiplicative)  $1, \rho, \dots, \rho^{p-2}$  of  $p-1$  elements. This group has in it just 10 elements or order dividing 10 viz.  $\rho^{j(p-1)/10}$  ( $j = 0, 1, \dots, 9$ ). These are represented (mod  $\pi$ ) by  $\pm 1, \pm \zeta, \dots, \pm \zeta^4$ , since these are distinct mod  $\pi$  and have order dividing 10. Now let  $\alpha$  be any non-zero residue mod  $\pi$ . Then  $\alpha^{(p-1)/10}$  has order dividing 10 and so is congruent to one of  $\pm 1, \pm \zeta, \dots, \pm \zeta^4 \pmod{\pi}$ . We define  $(\alpha/\pi)_{10} = \pm 1, \pm \zeta, \dots, \pm \zeta^4$  according as  $\alpha^{(p-1)/10}$  is congruent to  $\pm 1, \pm \zeta, \dots, \pm \zeta^4 \pmod{\pi}$ . It follows that

$$(\alpha/\pi)_{10} \equiv \alpha^{(N\pi-1)/10} \pmod{\pi}.$$

It is immediately verified that  $(\alpha\beta/\pi)_{10} = (\alpha/\pi)_{10} \cdot (\beta/\pi)_{10}$ , and we define  $(\alpha/\pi_1\pi_2)_{10} = (\alpha/\pi_1)_{10} \cdot (\alpha/\pi_2)_{10}$ . The following properties may be easily verified directly from the definition.

(i). If  $p \equiv 2, 3 \pmod{5}$ , so that  $p$  stays prime in  $\mathbf{Z}[\zeta]$ , and if  $n \in \mathbf{Z}$ , then  $(n/p)_{10} = 1$ .

(ii). If  $\pi$  is a prime factor of a  $p \equiv 4 \pmod{5}$ , so that  $p = \pi \bar{\pi}$  is the prime decomposition of  $p$  in  $\mathbf{Z}[\zeta]$ , and  $n \in \mathbf{Z}$ , then

$$(n/\pi)_{10} = 1.$$

(iii). If  $\pi$  is a prime factor of a  $p \equiv 1 \pmod{5}$ , so that  $p = \pi_1 \pi_2 \bar{\pi}_2 \bar{\pi}_1$  is the prime decomposition of  $p$  in  $\mathbf{Z}[\zeta]$ , then

$$(n/\pi)_{10} \cdot (n/\bar{\pi})_{10} = 1.$$

(iv). If  $\pi$  is a complex prime factor of a  $p \equiv 1, 4 \pmod{5}$  and  $\sigma$  of a  $q \equiv 1, 4 \pmod{5}$ , then  $\overline{(\pi/\sigma)_{10}} = (\bar{\pi}/\bar{\sigma})_{10}$ .

2. The symbol  $(\alpha/\beta)_5$  is defined in the same way and has similar properties.

3. The symbol  $(a/p)_{\mathbf{Z}}$  is simply the ordinary Legendre symbol, the subscript  $\mathbf{Z}$  is used to distinguish it from the symbol  $(\alpha/\beta)_2$  which denotes the quadratic character of  $\alpha$  modulo  $\beta$  in a given ring, e.g. if  $\alpha, \beta \in \mathbf{Z}[i]$

then  $(\alpha/\beta)_{\mathbf{Z}[i]} = \begin{cases} 1 & \text{if } x^2 \equiv \alpha \pmod{\beta} \text{ is solvable in } \mathbf{Z}[i], \\ -1 & \text{otherwise.} \end{cases}$

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