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Since the choice of g is arbitrary, we change g to another primitive root g^r with $(r, p-1) = 1$, $r = i \pmod{5}$, $i = 1, 2, 3, 4$. This does not alter Δ_a (as Δ_a is independent of g) but replaces π by any desired π_i so that $\Delta_a(\pi) = \Delta_a$ (any other π). Note that such an r exists, for all we want is, for $i = 1, 2, 3, 4$, a λ such that $(i+5\lambda, p-1) = 1$. Now $i+5\lambda$ takes infinitely many prime values as λ takes positive integer values since $(i, 5) = 1$; so λ may be chosen so that $i+5\lambda$ is a prime avoiding the primes occurring in $p-1$.

4. EXPRESSIONS ALLIED TO $\Delta_a(\pi)$

We fix our π now with $(g/\pi)_5 = \zeta$ and normalize it too. It is clear that there are only 3 expressions allied to $\Delta_a(\pi)$ viz $(-a/p)_Z (4a/\pi)_5 \cdot \pi \cdot \pi^\sigma +$ conjugates, $(-a/p)_Z (4a/\pi)_5 \cdot \pi^\sigma \cdot \pi^{\sigma^2} +$ conjugates and $(-a/p)_Z (4a/\pi)_5 \cdot \pi^{\sigma^2} \cdot \pi^{\sigma^3} +$ conjugates. This is so because changing the first term of $\Delta_a(\pi)$ fixes the changes in the other terms (otherwise we will not even get a rational integer!). Let us look at the first of these (the others would be similar), which equals $\text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \pi \pi^\sigma]$. We have the following theorem:

THEOREM 3. $\text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \pi \pi^\sigma] = \Delta_{au} - 1(\pi)$, where $(u/p)_Z = 1$ and $(u/\pi)_5 = (4a/\pi)_5$.

Proof. We have

$$\begin{aligned} \Delta_a(\pi) &= \text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \pi \cdot \pi^{\sigma^3}] \\ &= \text{Tr} [(-a/p)_Z (4a/\pi^\sigma)_5 \cdot \pi^\sigma \cdot \pi^{\sigma^3}] \text{ by 3 on letting } \pi \rightarrow \pi^\sigma, \\ &= \text{Tr} [(-a/p)_Z (16a^2/\pi)_5 \cdot \pi^\sigma \cdot \pi] \text{ since } (4a/\pi^\sigma)_5 = (g^v/\pi_2)_5 \\ &= (g^v/\pi_1)_5^2 = (4a/\pi)_5^2 = (16a^2/\pi)_5, \\ &= \text{Tr} [(-au/p)_Z (4(au)/\pi)_5 \cdot \pi \pi^\sigma], \text{ where } (u/p)_Z = 1 \text{ and } (u/p)_5 \\ &= (4a/\pi)_5. \end{aligned}$$

Now writing a for au we get the theorem.

It follows that the expressions allied to $\Delta_a(\pi)$ also represent the number of solutions of the congruence (1) for a suitable value of a .

5. THE SET $\{\Delta_a \mid a = 1, 2, 3, \dots, p-1\}$

Dickson's paper on cyclotomy [1] includes the following Theorem (theorem 8 of [1]). Let $p \equiv 1 \pmod{5}$ be a rational prime. Then the Diophantine equations

$$(4) \quad \begin{array}{l} \text{i. } 16p = x^2 + 50u^2 + 50v^2 + 125w^2 \\ \text{ii. } v^2 - 4uv - u^2 = xw \\ \text{iii. } x \equiv 1 \pmod{5} \end{array}$$

have exactly 4 integral simultaneous solutions. If (x, u, v, w) is one solution then the remaining three are $(x, -u, -v, w)$, $(x, v, -u, -w)$, $(x, -v, u, -w)$.

Now let $f(x, u, v, w) = \frac{1}{4}(25w - x - 10u - 20v)$. We have the following

THEOREM 4. *The distinct Δ_a are the following 10 numbers :*

$$\begin{array}{l} \pm x, \pm f(x, u, v, w), \pm f(x, -u, -v, w), \pm f(x, v, -u, -w), \\ \pm f(x, -v, u, -w). \end{array}$$

Remark. If $4a$ is a quintic residue mod p then $\Delta_a = (-a/p)_{\mathbf{Z}} \cdot x$.

Proof. In the notation of [2] we have

$$\Delta_a = (-a/p)_{\mathbf{Z}} \left[\left(\frac{4a}{\pi_1} \right)_5 \cdot T + \left(\frac{4a}{\pi_2} \right)_5 + S \cdot \left(\frac{4a}{\pi_3} \right)_5 \cdot \bar{S} + \left(\frac{4a}{\pi_4} \right)_5 \cdot \bar{T} \right]$$

with $T = s_1 \zeta + s_2 \zeta^2 + s_3 \zeta^3 + s_4 \zeta^4$ and $S = s_3 \zeta + s_1 \zeta^2 + s_4 \zeta^3 + s_2 \zeta^4$. Let $4a \equiv g^v \pmod{p}$. We have to look at the five cases $v \equiv 0, 1, 2, 3, 4 \pmod{5}$.

If $v \equiv 0 \pmod{5}$, so that $(4a/\pi_i)_5 = 1$ for all i , then

$$\begin{aligned} \Delta_a &= (-a/p)_{\mathbf{Z}} (T + \bar{T} + S + \bar{S}) = (-a/p)_{\mathbf{Z}} [(s_1 + s_4)(\zeta + \zeta^4) \\ &+ (s_2 + s_3)(\zeta^2 + \zeta^3) + (s_2 + s_3)(\zeta + \zeta^4) + (s_1 + s_4)(\zeta^2 + \zeta^3)] \\ &= (-a/p)_{\mathbf{Z}} [-(s_1 + s_2 + s_3 + s_4)] = (-a/p)_{\mathbf{Z}} \cdot x \text{ (see equation (62) of [1]).} \end{aligned}$$

If $v \equiv 1, 2, 3, 4 \pmod{5}$, we get respectively, as above

$$(5) \quad \Delta_a(\pi) = (-a/p)_{\mathbf{Z}} \begin{cases} 4s_4 - (s_1 + s_2 + s_3) & \text{if } v \equiv 1 \pmod{5}, \\ 4s_3 - (s_1 + s_2 + s_4) & \text{if } v \equiv 2 \pmod{5}, \\ 4s_2 - (s_1 + s_3 + s_4) & \text{if } v \equiv 3 \pmod{5}, \\ 4s_1 - (s_2 + s_3 + s_4) & \text{if } v \equiv 4 \pmod{5}. \end{cases}$$

Now from equations (62) and (63) of [1] we get, on solving

$$\begin{array}{l} 4s_1 = 5w - x + 2u + 4v, \\ 4s_2 = -5w - x + 4u - 2v, \\ 4s_3 = -5w - x - 4u + 2v, \\ 4s_4 = 5w - x - 2u - 4v. \end{array}$$

so that substitution in (5) gives

$$\Delta_a(\pi) = (-a/p)_Z \cdot \begin{cases} \frac{1}{4}(25w - x - 10u - 20v) & \text{if } v \equiv 1 \pmod{5}, \\ \frac{1}{4}(-25w - x - 20u + 10v) & \text{if } v \equiv 2 \pmod{5}, \\ \frac{1}{4}(-25w - x + 20u - 10v) & \text{if } v \equiv 3 \pmod{5}, \\ \frac{1}{4}(25w - x + 10u + 20v) & \text{if } v \equiv 4 \pmod{5}. \end{cases}$$

But letting $(x, u, v, w) \rightarrow (x, -u, -v, w), (x, v, -u, -w), (x, -v, u, -w)$ in the case $v \equiv 1 \pmod{5}$ gives just the cases $v \equiv 2, 3, 4 \pmod{5}$ respectively. This completes the proof of theorem 4.

6. A RELATION AND AN EXAMPLE

THEOREM 5. $(\Delta_g)^2 + (\Delta_{g^2})^2 + (\Delta_{g^3})^2 + (\Delta_{g^4})^2 + (\Delta_{g^5})^2 = 20 \cdot p$

Proof. The left hand side

$$\begin{aligned} &= [f(x, u, v, w)]^2 + [f(x, -u, -v, w)]^2 + \\ &\quad [f(x, v, -u, -w)]^2 + [f(x, -v, u, -w)]^2 + x^2 \\ &= \frac{1}{16} [4 \cdot 625w^2 + 4 \cdot x^2 + 1000(u^2 + v^2)] + x^2 \end{aligned}$$

on simplifying

$$\begin{aligned} &= \frac{5}{4}(125w^2 + x^2 + 50u^2 + 50v^2) = \frac{5}{4} \cdot 16 \cdot p \text{ (by } i \text{ of (4))} \\ &= 20 \cdot p \end{aligned}$$

as required.

An example. Let $p = 11$. The 4 solutions of (4) are

$$(1, 0, 1, 1), (1, 0, -1, 1), (1, 1, 0, -1), (1, -1, 0, -1)$$

and so by theorem 4 the set Δ_a is given by $\pm 1, \pm 4, -9, \pm 11, \pm 1$, so that $1^2 + 4^2 + 9^2 + 11^2 + 1^2 = 220 = 20 \cdot p$.