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$$\left(\frac{-4a}{\pi_1}\right)_{10} \cdot \pi_1 \pi_2 + \left(\frac{-4a}{\pi_2}\right)_{10} \cdot \pi_2 \pi_4 + \left(\frac{-4a}{\pi_3}\right)_{10} \cdot \pi_1 \pi_3 + \left(\frac{-4a}{\pi_4}\right)_{10} \cdot \pi_3 \pi_4$$

being the trace of  $(-4a/\pi_1)_{10} \cdot \pi_1 \pi_2$ , is a rational integer. What does it represent?

One could also remove the various restrictions on the  $\pi_i$  in the expression for  $\Delta_a$  and ask what  $\Delta_a$  then represents. The object of this note is to answer these questions and also to determine the set  $\{\Delta_a \mid a = 1, 2, 3, \dots, p-1\}$ .

It is immediate that  $\Delta_a$  can take only 10 distinct values. This follows by looking at (2) or directly from the congruence (1) as follows: Let  $(e, p) = 1$ , then we have

$$\Delta_a = \sum \left( \frac{x^5 - a}{p} \right) \text{ and so } \Delta_{ae} 5 = (e/p)_Z \cdot \Delta_a.$$

It follows that the distinct values taken by the  $\Delta_a$ , for  $a = 1, 2, \dots, p-1$  are just  $\pm \Delta_g, \pm \Delta_{g2}, \pm \Delta_{g3}, \pm \Delta_{g4}, \pm \Delta_{g5}$ . We shall determine these 10 values as a set. Which value is associated with which  $a$  will not be clear except when  $4a$  is a quintic residue mod  $p$ .

## 2. DETERMINATION OF $\Delta_a$ WITHOUT THE NORMALIZATION RESTRICTIONS ON THE $\pi_j$

Write  $p = \pi \cdot \pi^\sigma \cdot \pi^{\sigma^3} \cdot \pi^{\sigma^2}$  (with  $(g/\pi)_5 = \zeta = \pi_1 \pi_2 \pi_3 \pi_4$  say). Since the restrictions on  $\pi$  are going to be removed, we denote  $\Delta_a$  by  $\Delta_a(\pi)$ . We write (2) in a more convenient form viz

$$(3) \quad \Delta_a(\pi) = \left( \frac{-a}{p} \right)_Z \cdot \left[ \left( \frac{4a}{\pi_1} \right)_5 \cdot \pi_1 \pi_3 + \left( \frac{4a}{\pi_2} \right)_5 \cdot \pi_1 \pi_2 + \left( \frac{4a}{\pi_3} \right)_5 \cdot \pi_3 \pi_4 + \left( \frac{4a}{\pi_4} \right)_5 \cdot \pi_2 \pi_4 \right].$$

Thus  $\Delta_a(\pi) = \text{Tr} [(-a/p)_Z (4a/\pi)_5 \pi \pi^{\sigma^3}]$ .

Let the condition  $(g/\pi)_5 = \zeta$  be retained first so that we only change  $\pi$  to an associate  $\eta \pi$  where  $\eta = \zeta^i \varepsilon$  ( $0 \leq i \leq 4$ ) with  $\varepsilon$  a real fundamental

unit, say  $\pm \left( \frac{1 + \sqrt{5}}{2} \right)^j$ ,  $j \in \mathbf{Z}$ , of  $Q(\sqrt{5})$ . We have the following

**THEOREM 1.**  $\Delta_a(\zeta^i \varepsilon \cdot \pi) = \Delta_{ab}(\pi)$  where  $(b/\pi)_5 = \zeta^{5-i}$  and  $(b/p)_Z \neq N_{Q(\sqrt{5})/Q}(\varepsilon)$ .

*Proof.* Step 1.

$$\begin{aligned}\Delta_a(\zeta\pi) &= \text{Tr} [(-a/p)_Z (4a/\zeta\pi)_5 (\zeta\pi)(\zeta\pi)^{\sigma^3}] \\ &= \text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot \zeta^4 \cdot \pi\pi^{\sigma^3}] \\ &= \text{Tr} [(-au/p)_Z (4au/\pi)_5 \cdot \pi\pi^{\sigma^3}],\end{aligned}$$

where  $(u/p)_Z = 1$ ,  $(u/\pi)_5 = \zeta^4$ , and this  $= \Delta_{au}(\pi)$ . It follows that  $\Delta_a(\zeta^i\pi) = \Delta_{au}(\pi)$ , where  $(u/p)_Z = 1$  and  $(u/\pi)_5 = \zeta^{5-i}$  ( $i=0, 1, 2, 3, 4$ ).

Step 2.

$$\begin{aligned}\Delta_a(\varepsilon\pi) &= \text{Tr} [(-a/p)_Z (4a/\varepsilon\pi)_5 \cdot \varepsilon\pi \cdot (\varepsilon\pi)^{\sigma^3}] \\ &= \text{Tr} [(-a/p)_Z (4a/\pi)_5 \cdot N_{Q(\sqrt{5})/Q}(\varepsilon) \cdot \pi\pi^{\sigma^3}] \\ &= \Delta_{av}(\pi),\end{aligned}$$

where  $(v/p)_Z = N_{Q(\sqrt{5})/Q}(\varepsilon)$ ,  $(v/\pi)_5 = 1$ .

Combining steps 1 and 2 we get:

$$\begin{aligned}\Delta_a(\zeta^i\varepsilon\pi) &= \Delta_{au}(\varepsilon\pi) \text{ where } (u/p)_Z = 1, (u/\pi)_5 = \zeta^{5-i} \\ &= \Delta_{au.v}(\pi) \text{ where } (v/p)_Z = \text{Norm } \varepsilon, (v/\pi)_5 = 1, \\ &= \Delta_{ab}(\pi) \text{ where } b = uv \text{ satisfies the conditions of theorem 1.}\end{aligned}$$

This completes the proof of theorem 1.

We next remove the restriction  $(g/\pi)_5 = \zeta$  and see what the  $\Delta_a$ 's mean then.

### 3. THE RESTRICTION $(g/\pi)_5 = \zeta$ REMOVED

Here we have to look at  $\Delta_a(\pi^\sigma)$  (and similarly  $\Delta_a(\pi^{\sigma^2})$  and  $\Delta_a(\pi^{\sigma^3})$ ). We have the following

**THEOREM 2.**  $\Delta_a(\pi^\sigma) = \Delta_a(\pi)$ .

*Proof.*  $\Delta_a(\pi^\sigma) = \text{Tr} [(-a/p)_Z (4a/\pi^\sigma)_5 \cdot \pi^\sigma \cdot (\pi^\sigma)^{\sigma^3}]$ .

Now  $(4a/\pi^\sigma)_5 = (4a/\pi_2)_5$ , and if  $4a \equiv g^v \pmod{p}$  then this  $= (g^v/\pi_2)_5 = (g/\pi_2)_5^v = \zeta^{2v} = (g^v/\pi_1)_5^2 = (4a/\pi_1)_5^2 = \sigma[(4a/\pi)_5]$ . Hence

$$\begin{aligned}\Delta_a(\pi^\sigma) &= \text{Tr} [(-a/p)_Z \cdot \sigma(4a/\pi)_5 \cdot \pi \cdot \pi^{\sigma^3}] \\ &= \text{Tr} [\sigma((-a/p)_Z (4a/\pi)_5 \cdot \pi\pi^{\sigma^3})] \\ &= \Delta_a(\pi) \text{ as required.}\end{aligned}$$

A clearer insight is gained into this by looking at the whole thing directly as follows.