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# NOTES ON THE CONGRUENCE $y^2 \equiv x^5 - a \pmod{p}$

by A. R. RAJWADE

## 1. INTRODUCTION

In a previous paper [3] we proved the following

**THEOREM.** *Let  $p \equiv 1 \pmod{5}$  be a rational prime and  $g$  a fixed primitive root mod  $p$ . Then the number of solutions of the congruence*

$$(1) \quad y^2 \equiv x^5 - a \pmod{p}$$

is  $p + \Delta_a$ , where  $\Delta_a$  is equal to <sup>1)</sup>

$$(2) \quad \left(\frac{-4a}{\pi_1}\right)_{10} \cdot \pi_3 \pi_4 + \left(\frac{-4a}{\pi_2}\right)_{10} \cdot \pi_1 \pi_3 \\ + \left(\frac{-4a}{\pi_3}\right)_{10} \cdot \pi_2 \pi_4 + \left(\frac{-4a}{\pi_4}\right)_{10} \cdot \pi_1 \pi_2 .$$

Here  $p = \pi_1 \pi_2 \pi_3 \pi_4 = \pi_1 \cdot \sigma \pi_1 \cdot \sigma^3 \pi_1 \cdot \sigma^2 \pi_1$ , with  $\sigma: \zeta \rightarrow \zeta^2$ , is the decomposition of  $p$  in  $Q(\zeta)$ ,  $\zeta^5 = 1$ ,  $\zeta \neq 1$  and  $\pi_1$  is chosen to satisfy  $(g/\pi_1)_5 = \zeta$ , so that  $(g/\pi_i)_5 = \zeta^i$ , and the  $\pi_j$  are normalized so that the products  $S = \pi_1 \pi_2$ ,  $\bar{S} = \pi_3 \pi_4$ ,  $T = \pi_1 \pi_3$ ,  $\bar{T} = \pi_2 \pi_4$  (all polynomials in  $\zeta$ ) satisfy

1.  $S(\zeta) \cdot S(\zeta^{-1}) \equiv [S(1)]^2 \pmod{5}$ ,
2.  $S(\zeta) \equiv S(1) \pmod{(1-\zeta)^2}$ ,
3.  $S(1) \equiv 4 \pmod{5}$ .

(and similarly for  $\bar{S}, T, \bar{T}$ ).

In (2) the 4 products  $\pi_i \pi_j$  are those 4 out of the 6 combinations  $\pi_1 \pi_2, \pi_1 \pi_3, \pi_1 \pi_4, \pi_2 \pi_3, \pi_2 \pi_4, \pi_3 \pi_4$  for which  $\bar{\pi}_i \neq \pi_j$ . But there is no symmetrical way of coupling the residue symbol  $\left(\frac{-4a}{\pi_i}\right)_{10}$  with  $\pi_j \pi_k$ . We ask: What do other expressions similar to  $\Delta_a$  represent? For example the expression

<sup>1)</sup> See Appendix for the definitions of  $(\alpha' \beta)_{10}, (\alpha' \beta)_5, (a/p)_Z$ .

$$\left(\frac{-4a}{\pi_1}\right)_{10} \cdot \pi_1 \pi_2 + \left(\frac{-4a}{\pi_2}\right)_{10} \cdot \pi_2 \pi_4 + \left(\frac{-4a}{\pi_3}\right)_{10} \cdot \pi_1 \pi_3 + \left(\frac{-4a}{\pi_4}\right)_{10} \cdot \pi_3 \pi_4$$

being the trace of  $(-4a/\pi_1)_{10} \cdot \pi_1 \pi_2$ , is a rational integer. What does it represent?

One could also remove the various restrictions on the  $\pi_i$  in the expression for  $\Delta_a$  and ask what  $\Delta_a$  then represents. The object of this note is to answer these questions and also to determine the set  $\{\Delta_a \mid a = 1, 2, 3, \dots, p - 1\}$ .

It is immediate that  $\Delta_a$  can take only 10 distinct values. This follows by looking at (2) or directly from the congruence (1) as follows: Let  $(e, p) = 1$ , then we have

$$\Delta_a = \sum \left( \frac{x^5 - a}{p} \right) \text{ and so } \Delta_{ae} 5 = (e/p)_Z \cdot \Delta_a.$$

It follows that the distinct values taken by the  $\Delta_a$ , for  $a = 1, 2, \dots, p - 1$  are just  $\pm \Delta_g, \pm \Delta_{g^2}, \pm \Delta_{g^3}, \pm \Delta_{g^4}, \pm \Delta_{g^5}$ . We shall determine these 10 values as a set. Which value is associated with which  $a$  will not be clear except when  $4a$  is a quintic residue mod  $p$ .

## 2. DETERMINATION OF $\Delta_a$

WITHOUT THE NORMALIZATION RESTRICTIONS ON THE  $\pi_j$

Write  $p = \pi \cdot \pi^\sigma \cdot \pi^{\sigma^3} \cdot \pi^{\sigma^2}$  (with  $(g/\pi)_5 = \zeta) = \pi_1 \pi_2 \pi_3 \pi_4$  say. Since the restrictions on  $\pi$  are going to be removed, we denote  $\Delta_a$  by  $\Delta_a(\pi)$ . We write (2) in a more convenient form viz

$$(3) \quad \Delta_a(\pi) = \left(\frac{-a}{p}\right)_Z \cdot \left[ \left(\frac{4a}{\pi_1}\right)_5 \cdot \pi_1 \pi_3 + \left(\frac{4a}{\pi_2}\right)_5 \cdot \pi_1 \pi_2 + \left(\frac{4a}{\pi_3}\right)_5 \cdot \pi_3 \pi_4 + \left(\frac{4a}{\pi_4}\right)_5 \cdot \pi_2 \pi_4 \right].$$

Thus  $\Delta_a(\pi) = \text{Tr} [(-a/p)_Z (4a/\pi)_5 \pi \pi^{\sigma^3}]$ .

Let the condition  $(g/\pi)_5 = \zeta$  be retained first so that we only change  $\pi$  to an associate  $\eta \pi$  where  $\eta = \zeta^i \varepsilon$  ( $0 \leq i \leq 4$ ) with  $\varepsilon$  a real fundamental unit, say  $\pm \left(\frac{1 + \sqrt{5}}{2}\right)^j$ ,  $j \in \mathbf{Z}$ , of  $Q(\sqrt{5})$ . We have the following

**THEOREM 1.**  $\Delta_a(\zeta^i \varepsilon \cdot \pi) = \Delta_{ab}(\pi)$  where  $(b/\pi)_5 = \zeta^{5-i}$  and  $(b/p)_Z \neq N_{Q(\sqrt{5})/Q}(\varepsilon)$ .