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At the end of the proof of Theorem 2 we showed that $\int_{-\infty}^{\infty} K_T(u)\, du = 1.$

Also, (2) implies that $K_T(x) \geqslant 0$ for all x. Now formula (8) can be interpreted as expressing a certain average of $x^{-\frac{1}{2}} \{\Pi(x) - \tau(x)\} \log x$ as $\Sigma(x)$ plus a bounded error term.

It follows that there exist sequences $\{x_n\}$ and $\{y_n\}$ tending to infinity for which

$$x_n^{-\frac{1}{2}} \{\Pi(x_n) - \tau(x_n)\} \log x_n > c$$
$$y_n^{-\frac{1}{2}} \{\Pi(y_n) - \tau(y_n)\} \log y_n < -c$$

for any given number $c > 0$. Thus

$$x^{-\frac{1}{2}} \{\Pi(x) - \tau(x)\} \log x$$

is unbounded from above and below. If we recall (3), we have completed our proof that $\pi(x) - \operatorname{li} x$ changes sign infinitely often.

4. Further results

Littlewood actually showed a bit more than we have. He proved that

$$x^{-\frac{1}{2}} \{\pi(x) - \operatorname{li} x\} \log x / \log\log\log x$$

has a positive limit superior and negative limit inferior. The best account of this estimate is probably that given in [5].

It appears that our arguments can be extended to achieve this estimate. The contradiction arguments can be reorganized, exploiting more fully the hypothesized one sided bound in Theorem 2. However, we would also require an explicit estimate in place of the $o_T(1)$ in the conclusion of this theorem. Such estimation would cancel out the economy we have achieved.

It is reasonable to ask for an $x > 3/2$ for which $\pi(x) - \operatorname{li} x > 0$. The first person to provide an estimate of such a number x was Skewes [13]. He showed that there exists an $x < \exp\exp\exp\exp(7.705)$ for which $\pi(x) - \operatorname{li} x > 0$. This enormous bound was reduced to a more modest $1.65 \cdot 10^{1165} \approx \exp\exp(7.895)$ by R. S. Lehman [10]. Each of these authors combined theoretical arguments with extensive numerical calculations using the position of many zeros of the Riemann zeta function. The case in which the Riemann hypothesis is assumed false requires much more work than we had to do.

It would be interesting to know explicitly the first (or indeed any!) $x > 3/2$ for which $\pi(x) - \operatorname{li} x > 0$. Lehman observed in [10] that it seemed likely that such a number would have to exceed 10^{20}.

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