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CHANGES OF SIGN OF $\pi(x) - \operatorname{li} x$

by Harold G. Diamond [1]

1. Introduction

The prime number theorem asserts that $\pi(x)$, the number of primes not exceeding x, is asymptotic to

$$\operatorname{li} x \overset{def}{=} \lim_{\varepsilon \to 0+} \left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^{x} \right) \frac{du}{\log u}$$

as $x \to \infty$. It has been shown [12, p. 72] that $\pi(x) < \operatorname{li} x$ for $3/2 \leqslant x \leqslant 10^8$, and it was once conjectured that this inequality prevailed for all $x \geqslant 3/2$. However, this conjecture was disproved by Littlewood [11] who established

Theorem 1. $\pi(x) - \operatorname{li} x$ *changes sign infinitely often as* $x \to \infty$.

Littlewood's proof was simplified by Ingham [5]. In the present article we make a further simplification by eliminating use of the so called explicit formula for ψ (cf. [4], pp. 76-80). The deepest fact which we require from analytic number theory is an estimate of the size of $N(T)$, the number of zeros ρ of the Riemann zeta function satisfying $0 < \operatorname{Im} \rho \leqslant T$.

The key step in the argument is Theorem 2, which is given in the next section. This result, which is based on another article of Ingham [6], enables us to relate a certain average of the function π to zeros of the Riemann zeta function.

2. A tauberian theorem

We begin by giving an extension of the Wiener-Ikehara tauberian theorem. Our result admits poles and certain "lesser" singularities on the abscissa of convergence of the transformed function. We adhere to the curious convention of expressing the complex variable s as $\sigma + it$.

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