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# ON TRANSLATIVE SUBDIVISIONS OF CONVEX DOMAINS 

by H. Groemer ${ }^{1}$ )

In euclidean $n$-space $R^{n}$ let $K$ be a convex body (compact convex subset of $R^{n}$ with interior points). Let $\mathbf{S}=\left\{S_{1}, S_{2}, \ldots S_{m}\right\}$ be a finite collection of at least two closed subsets of $K$ such that each $S_{i}$ can be obtained from any $S_{j}$ by a translation. Then, $\mathbf{S}$ will be called a translative subdivision of $K$ if

$$
\begin{equation*}
S_{1} \cup S_{2} \cup \ldots \cup S_{m}=K \tag{1}
\end{equation*}
$$

and if for $i \neq j$

$$
\operatorname{int} S_{i} \cap \operatorname{int} S_{j}=\varnothing
$$

Under the assumption that the sets $S_{i}$ of a translative subdivision of a convex body $K$ are also convex it can be shown that $K$ and the sets $S_{i}$ must be cylinders (for $n=2$ parallelograms). Also, the possible arrangements of the sets $S_{i}$ can be completely described (see [2]). Related to this result is the question whether there exist a convex body $K$ and a translative subdivision $\left\{S_{1}, S_{2}, \ldots S_{m}\right\}$ of $K$ with sets $S_{i}$ that are not convex. If no assumptions concerning the regularity or connectivity of the sets $S_{i}$ are made, there are trivial examples of convex bodies (e.g. cubes) which permit such non-convex subdivisions. To obtain a meaningful problem let us call a subset $M$ of $R^{n}$ strongly connected if any two of its points can be connected in the interior of $M$ by a Jordan arc; that means, if $x \in M, y \in M, x \neq y$ there exists a Jordan arc $\gamma$ with $x$ and $y$ as endpoints and such that every point of $\gamma$ which is different from $x$ and $y$ is contained in the interior of $M$. Using this definition, the question can be raised whether there exists a convex body with a translative subdivision that consist of strongly connected non-convex sets. For $n=1$ the situation is completely trivial. For $n \geqq 3$ this problem has not yet been solved. In the present paper the case $n=2$ is settled by the following theorem which will be proved with the aid of the Jordan curve theorem. As a convenient abbreviation a two-dimensional convex body will be called a convex domain.

[^0]Theorem. If a translative subdivision of a convex domain consists of strongly connected compact sets, then these sets are necessarily convex (and therefore parallelograms).

Proof: Let $K$ be a given convex domain and let us assume that $K$ has a translative subdivision $\left\{S_{1}, S_{2}, \ldots S_{m}\right\}$ with strongly connected nonconvex sets $S_{i}$. As a notational simplification, the set $S_{1}$ will often be denoted by $S$. Now there are two possibilities. Either the boundary of the convex hull of $S$ is contained in $S$ or this is not the case.
I. Assume that

$$
\begin{equation*}
\text { bdr conv } S \subset S \tag{3}
\end{equation*}
$$

where conv $S$ denotes the convex hull of $S$. Since $S$ is not convex there is a point $p$ with $p \in \operatorname{conv} S$ and

$$
\begin{equation*}
p \notin S \tag{4}
\end{equation*}
$$

Because of conv $S=$ bdr conv $S \cup$ int conv $S$ and (3) this implies

$$
\begin{equation*}
p \in \operatorname{int} \operatorname{conv} S \tag{5}
\end{equation*}
$$

From the convexity of $K$ and $S \subset K$ it follows that conv $S \subset K$ and therefore

$$
\begin{equation*}
p \in K . \tag{6}
\end{equation*}
$$

The relations (1), (4), and (6) imply

$$
\begin{equation*}
p \in S_{j}=S+t \tag{7}
\end{equation*}
$$

for some $j \neq 1$ and a translation vector $t \neq 0$. The set $S+t$ is not contained in conv $S$ (for this would imply (conv $S$ ) $+t=\operatorname{conv}(S+t) \subset \operatorname{conv} S$ which is clearly impossible since a convex domain cannot contain a translate of itself). Hence, there is a point $q$ with

$$
\begin{equation*}
q \notin \operatorname{conv} S \tag{8}
\end{equation*}
$$

$$
\begin{equation*}
q \in S+t \tag{9}
\end{equation*}
$$

(7) and (9) show that $p$ and $q$ can be connected in the interior of $S+t$ by some Jordan arc $\gamma$. From (5) and (8) one obtains that $\gamma$ has a point, say $x$, in common with bdr conv $S$. Because of the assumption (3) it is clear that

$$
\begin{equation*}
x \in S \tag{10}
\end{equation*}
$$

On the other hand, (5) and (8) show that $x \neq p, x \neq q$ and therefore

$$
\begin{equation*}
x \in \operatorname{int}(S+t)=\operatorname{int} S_{j} \tag{11}
\end{equation*}
$$

Because of (10) and the strong connectivity of $S$ there are interior points of $S$ in any neighborhood of $x$. This together with (11) shows that for some $j \neq 1$

$$
\operatorname{int} S_{1} \cap \operatorname{int} S_{j} \neq \varnothing
$$

in contradiction to (2).
II. Assume that bdr conv $S \not \ddagger S$. This means that there exists a point $g$ with $g \in \operatorname{bdr}$ conv $S$ and

$$
\begin{equation*}
g \notin S \tag{12}
\end{equation*}
$$

By a well-known version of the theorem of Carathéodory on the convex hull of connected sets (see Bonnesen-Fenchel [1], p. 9) there is a closed line segment $\sigma=\left[s_{1}, s_{2}\right]$ with $s_{1} \in S, s_{2} \in S$ and $g$ in its (relative) interior. If $L$ denotes a support line for conv $S$ which contains $g$, it is obvious that

$$
\begin{equation*}
\sigma \subset L . \tag{13}
\end{equation*}
$$

Let $H$ be the halfplane which is bounded by $L$ and contains conv $S$, and let $H_{i}$ be defined by $H_{i}=H+t_{i}$ where $t_{i}$ is the translation vector determined by $S_{i}=S+t_{i}$. Then, the union of all the halfplanes $H_{i}$ is again one of these halfplanes, say $H_{k}$. Since $H_{k}$ contains every $S_{i}$ it follows that the line $L_{k}=L+t_{k}$ is a support line of $K$. By a proper assignment of the subscripts it can be achieved that $k=1$ and therefore $L_{k}=L$. Hence, there is no loss in generality by assuming that the line $L$ which contains $\sigma$ is a support line of $K$. This implies in particular that

$$
\begin{equation*}
\sigma \subset \operatorname{bdr} K \tag{14}
\end{equation*}
$$

Because of the strong connectivity of $S$ it is possible to connect the points $s_{1}$ and $s_{2}$ in the interior of $S$ by some Jordan arc $\tau$. Since (13) implies that $\sigma$ contains no interior points of $S$ the arcs $\sigma$ and $\tau$ have only the points $s_{1}$ and $s_{2}$ in common. Let $\lambda$ be the closed Jordan curve composed of $\sigma$ and $\tau$. Then, the Jordan curve theorem shows that the complement of $\lambda$ (with respect to $R^{2}$ ) consists of two open connected sets which have the same boundary, namely $\lambda$. Further, one of these regions, say $J$, is bounded and the other is unbounded.

From the inclusions $J \subset(J \cup \lambda) \subset \operatorname{conv}(J \cup \lambda)=\operatorname{conv} \lambda$ and $\lambda=(\sigma \cup \tau) \subset((\operatorname{conv} S) \cup S) \subset \operatorname{conv} S \subset K$ it follows immediately that

$$
\begin{equation*}
J \subset \operatorname{conv} S \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
J \subset K \tag{16}
\end{equation*}
$$

Because of (12) and the compactness of $S$ it is obvious that the point $g$ has positive distance from $S$. Using the fact that $g \in \lambda=\operatorname{bdr} J$ one can find a point $q$ in $J$ which is so close to $g$ that $q \notin S$. This, together with (1) and (16) shows that $q$ is contained in some $S_{h} \neq S$. Actually, one may assume that $q$ is in the interior of $S_{h}$. If necessary this can be achieved by a sufficiently small change in the selection of $q$ without disturbing the relations $q \in J, q \notin S$. On the other hand, there is a point $p$ with $p \in \operatorname{int} S_{h}$ and $p \notin J$. If such a point would not exist one had int $S_{h} \subset J$. But then (15) shows that int $S_{h} \subset \operatorname{conv} S$ and by taking the closure and the convex hull one would obtain conv $S_{h}=(\operatorname{conv} S)+t_{h} \subset \operatorname{conv} S$ with $t_{h} \neq 0$ and this is certainly impossible. Note that the closure of the interior of $S_{h}$ is $S_{h}$ since the strong connectivity implies that there are interior points in any neighborhood of a boundary point.

Hence, it has been found that there are points $p, q$ with the following properties:

$$
\begin{gather*}
p \in \operatorname{int} S_{h}, q \in \operatorname{int} S_{h},  \tag{17}\\
p \notin J, q \in J . \tag{18}
\end{gather*}
$$

Let $\kappa$ be a Jordan arc which connects $p$ and $q$ in the interior of $S_{h}$. Because of (17) the endpoints of $\kappa$ are also in int $S_{h}$ and therefore in int $K$. This fact, if compared with (14), shows that $\kappa$ and $\sigma$ are disjoint. On the other hand, it follows from (18) that $\kappa$ must contain a point of bdr $J=\lambda=\tau \cup \sigma$. Writing $\tau^{\prime}=\tau-\left\{s_{1}, s_{2}\right\}$ it has therefore been shown that $\kappa$ and $\tau^{\prime}$ have a point, say $x$, in common. But this implies that $x \in \kappa \subset$ int $S_{h}$ and $x \in \tau^{\prime} \subset \operatorname{int} S_{1}$, which contradicts the assumption (2).

## REFERENCES

[1] Bonnesen, T. und W. Fenchel. Theorie der konvexen Körper. Ergebn. d. Math., Bd. 3. Berlin-Göttingen-Heidelberg, Springer 1934.
[2] Groemer, H. Über translative Zerlegungen konvexer Körper. Arch. d. Math. 19 (1968), 445-448.
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