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# ON TRANSLATIVE SUBDIVISIONS OF CONVEX DOMAINS

by H. GROEMER <sup>1)</sup>

In Euclidean  $n$ -space  $R^n$  let  $K$  be a convex body (compact convex subset of  $R^n$  with interior points). Let  $S = \{S_1, S_2, \dots, S_m\}$  be a finite collection of at least two closed subsets of  $K$  such that each  $S_i$  can be obtained from any  $S_j$  by a translation. Then,  $S$  will be called a *translative subdivision* of  $K$  if

$$(1) \quad S_1 \cup S_2 \cup \dots \cup S_m = K,$$

and if for  $i \neq j$

$$\text{int } S_i \cap \text{int } S_j = \emptyset.$$

Under the assumption that the sets  $S_i$  of a translative subdivision of a convex body  $K$  are also convex it can be shown that  $K$  and the sets  $S_i$  must be cylinders (for  $n=2$  parallelograms). Also, the possible arrangements of the sets  $S_i$  can be completely described (see [2]). Related to this result is the question whether there exist a convex body  $K$  and a translative subdivision  $\{S_1, S_2, \dots, S_m\}$  of  $K$  with sets  $S_i$  that are not convex. If no assumptions concerning the regularity or connectivity of the sets  $S_i$  are made, there are trivial examples of convex bodies (e.g. cubes) which permit such non-convex subdivisions. To obtain a meaningful problem let us call a subset  $M$  of  $R^n$  *strongly connected* if any two of its points can be connected in the interior of  $M$  by a Jordan arc; that means, if  $x \in M, y \in M, x \neq y$  there exists a Jordan arc  $\gamma$  with  $x$  and  $y$  as endpoints and such that every point of  $\gamma$  which is different from  $x$  and  $y$  is contained in the interior of  $M$ . Using this definition, the question can be raised whether there exists a convex body with a translative subdivision that consist of strongly connected non-convex sets. For  $n = 1$  the situation is completely trivial. For  $n \geq 3$  this problem has not yet been solved. In the present paper the case  $n = 2$  is settled by the following theorem which will be proved with the aid of the Jordan curve theorem. As a convenient abbreviation a two-dimensional convex body will be called a convex domain.

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**THEOREM.** *If a translative subdivision of a convex domain consists of strongly connected compact sets, then these sets are necessarily convex (and therefore parallelograms).*

**PROOF:** Let  $K$  be a given convex domain and let us assume that  $K$  has a translative subdivision  $\{S_1, S_2, \dots, S_m\}$  with strongly connected non-convex sets  $S_i$ . As a notational simplification, the set  $S_1$  will often be denoted by  $S$ . Now there are two possibilities. Either the boundary of the convex hull of  $S$  is contained in  $S$  or this is not the case.

I. Assume that

$$(3) \quad \text{bdr conv } S \subset S,$$

where  $\text{conv } S$  denotes the convex hull of  $S$ . Since  $S$  is not convex there is a point  $p$  with  $p \in \text{conv } S$  and

$$(4) \quad p \notin S.$$

Because of  $\text{conv } S = \text{bdr conv } S \cup \text{int conv } S$  and (3) this implies

$$(5) \quad p \in \text{int conv } S.$$

From the convexity of  $K$  and  $S \subset K$  it follows that  $\text{conv } S \subset K$  and therefore

$$(6) \quad p \in K.$$

The relations (1), (4), and (6) imply

$$(7) \quad p \in S_j = S + t$$

for some  $j \neq 1$  and a translation vector  $t \neq 0$ . The set  $S + t$  is not contained in  $\text{conv } S$  (for this would imply  $(\text{conv } S) + t = \text{conv } (S+t) \subset \text{conv } S$  which is clearly impossible since a convex domain cannot contain a translate of itself). Hence, there is a point  $q$  with

$$(8) \quad q \notin \text{conv } S,$$

$$(9) \quad q \in S + t.$$

(7) and (9) show that  $p$  and  $q$  can be connected in the interior of  $S + t$  by some Jordan arc  $\gamma$ . From (5) and (8) one obtains that  $\gamma$  has a point, say  $x$ , in common with  $\text{bdr conv } S$ . Because of the assumption (3) it is clear that

$$(10) \quad x \in S.$$

On the other hand, (5) and (8) show that  $x \neq p$ ,  $x \neq q$  and therefore

$$(11) \quad x \in \text{int}(S + t) = \text{int } S_j.$$

Because of (10) and the strong connectivity of  $S$  there are interior points of  $S$  in any neighborhood of  $x$ . This together with (11) shows that for some  $j \neq 1$

$$\text{int } S_1 \cap \text{int } S_j \neq \emptyset$$

in contradiction to (2).

II. Assume that  $\text{bdr conv } S \notin S$ . This means that there exists a point  $g$  with  $g \in \text{bdr conv } S$  and

$$(12) \quad g \notin S.$$

By a well-known version of the theorem of Carathéodory on the convex hull of connected sets (see Bonnesen-Fenchel [1], p. 9) there is a closed line segment  $\sigma = [s_1, s_2]$  with  $s_1 \in S$ ,  $s_2 \in S$  and  $g$  in its (relative) interior. If  $L$  denotes a support line for  $\text{conv } S$  which contains  $g$ , it is obvious that

$$(13) \quad \sigma \subset L.$$

Let  $H$  be the halfplane which is bounded by  $L$  and contains  $\text{conv } S$ , and let  $H_i$  be defined by  $H_i = H + t_i$  where  $t_i$  is the translation vector determined by  $S_i = S + t_i$ . Then, the union of all the halfplanes  $H_i$  is again one of these halfplanes, say  $H_k$ . Since  $H_k$  contains every  $S_i$  it follows that the line  $L_k = L + t_k$  is a support line of  $K$ . By a proper assignment of the subscripts it can be achieved that  $k = 1$  and therefore  $L_k = L$ . Hence, there is no loss in generality by assuming that the line  $L$  which contains  $\sigma$  is a support line of  $K$ . This implies in particular that

$$(14) \quad \sigma \subset \text{bdr } K.$$

Because of the strong connectivity of  $S$  it is possible to connect the points  $s_1$  and  $s_2$  in the interior of  $S$  by some Jordan arc  $\tau$ . Since (13) implies that  $\sigma$  contains no interior points of  $S$  the arcs  $\sigma$  and  $\tau$  have only the points  $s_1$  and  $s_2$  in common. Let  $\lambda$  be the closed Jordan curve composed of  $\sigma$  and  $\tau$ . Then, the Jordan curve theorem shows that the complement of  $\lambda$  (with respect to  $R^2$ ) consists of two open connected sets which have the same boundary, namely  $\lambda$ . Further, one of these regions, say  $J$ , is bounded and the other is unbounded.

From the inclusions  $J \subset (J \cup \lambda) \subset \text{conv}(J \cup \lambda) = \text{conv } \lambda$  and  $\lambda = (\sigma \cup \tau) \subset ((\text{conv } S) \cup S) \subset \text{conv } S \subset K$  it follows immediately that

$$(15) \quad J \subset \text{conv } S$$

and

$$(16) \quad J \subset K.$$

Because of (12) and the compactness of  $S$  it is obvious that the point  $g$  has positive distance from  $S$ . Using the fact that  $g \in \lambda = \text{bdr } J$  one can find a point  $q$  in  $J$  which is so close to  $g$  that  $q \notin S$ . This, together with (1) and (16) shows that  $q$  is contained in some  $S_h \neq S$ . Actually, one may assume that  $q$  is in the interior of  $S_h$ . If necessary this can be achieved by a sufficiently small change in the selection of  $q$  without disturbing the relations  $q \in J$ ,  $q \notin S$ . On the other hand, there is a point  $p$  with  $p \in \text{int } S_h$  and  $p \notin J$ . If such a point would not exist one had  $\text{int } S_h \subset J$ . But then (15) shows that  $\text{int } S_h \subset \text{conv } S$  and by taking the closure and the convex hull one would obtain  $\text{conv } S_h = (\text{conv } S) + t_h \subset \text{conv } S$  with  $t_h \neq 0$  and this is certainly impossible. Note that the closure of the interior of  $S_h$  is  $S_h$  since the strong connectivity implies that there are interior points in any neighborhood of a boundary point.

Hence, it has been found that there are points  $p, q$  with the following properties:

$$(17) \quad p \in \text{int } S_h, \quad q \in \text{int } S_h,$$

$$(18) \quad p \notin J, \quad q \in J.$$

Let  $\kappa$  be a Jordan arc which connects  $p$  and  $q$  in the interior of  $S_h$ . Because of (17) the endpoints of  $\kappa$  are also in  $\text{int } S_h$  and therefore in  $\text{int } K$ . This fact, if compared with (14), shows that  $\kappa$  and  $\sigma$  are disjoint. On the other hand, it follows from (18) that  $\kappa$  must contain a point of  $\text{bdr } J = \lambda = \tau \cup \sigma$ . Writing  $\tau' = \tau - \{s_1, s_2\}$  it has therefore been shown that  $\kappa$  and  $\tau'$  have a point, say  $x$ , in common. But this implies that  $x \in \kappa \subset \text{int } S_h$  and  $x \in \tau' \subset \text{int } S_1$ , which contradicts the assumption (2).

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