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# ON TRANSLATIVE SUBDIVISIONS OF CONVEX DOMAINS

by H. Groemer 1)

In euclidean *n*-space  $R^n$  let K be a convex body (compact convex subset of  $R^n$  with interior points). Let  $S = \{S_1, S_2, ... S_m\}$  be a finite collection of at least two closed subsets of K such that each  $S_i$  can be obtained from any  $S_j$  by a translation. Then, S will be called a *translative subdivision* of K if

$$(1) S_1 \cup S_2 \cup \ldots \cup S_m = K,$$

and if for  $i \neq j$ 

int 
$$S_i \cap \text{int } S_i = \emptyset$$
.

Under the assumption that the sets  $S_i$  of a translative subdivision of a convex body K are also convex it can be shown that K and the sets  $S_i$  must be cylinders (for n=2 parallelograms). Also, the possible arrangements of the sets  $S_i$  can be completely described (see [2]). Related to this result is the question whether there exist a convex body K and a translative subdivision  $\{S_1, S_2, ... S_m\}$  of K with sets  $S_i$  that are not convex. If no assumptions concerning the regularity or connectivity of the sets  $S_i$  are made, there are trivial examples of convex bodies (e.g. cubes) which permit such non-convex subdivisions. To obtain a meaningful problem let us call a subset M of  $R^n$ strongly connected if any two of its points can be connected in the interior of M by a Jordan arc; that means, if  $x \in M$ ,  $y \in M$ ,  $x \neq y$  there exists a Jordan arc y with x and y as endpoints and such that every point of y which is different from x and y is contained in the interior of M. Using this definition, the question can be raised whether there exists a convex body with a translative subdivision that consist of strongly connected non-convex sets. For n = 1 the situation is completely trivial. For  $n \ge 3$  this problem has not yet been solved. In the present paper the case n=2 is settled by the following theorem which will be proved with the aid of the Jordan curve theorem. As a convenient abbreviation a two-dimensional convex body will be called a convex domain.

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THEOREM. If a translative subdivision of a convex domain consists of strongly connected compact sets, then these sets are necessarily convex (and therefore parallelograms).

PROOF: Let K be a given convex domain and let us assume that K has a translative subdivision  $\{S_1, S_2, ... S_m\}$  with strongly connected non-convex sets  $S_i$ . As a notational simplification, the set  $S_i$  will often be denoted by S. Now there are two possibilities. Either the boundary of the convex hull of S is contained in S or this is not the case.

## I. Assume that

$$(3) bdr conv S \subset S,$$

where conv S denotes the convex hull of S. Since S is not convex there is a point p with  $p \in \text{conv } S$  and

$$(4) p \notin S.$$

Because of conv  $S = bdr conv S \cup int conv S$  and (3) this implies

(5) 
$$p \in \text{int conv } S$$
.

From the convexity of K and  $S \subset K$  it follows that conv  $S \subset K$  and therefore

$$(6) p \in K.$$

The relations (1), (4), and (6) imply

$$(7) p \in S_i = S + t$$

for some  $j \neq 1$  and a translation vector  $t \neq 0$ . The set S + t is not contained in conv S (for this would imply (conv S) +  $t = \text{conv}(S+t) \subset \text{conv } S$  which is clearly impossible since a convex domain cannot contain a translate of itself). Hence, there is a point q with

$$(8) q \notin \operatorname{conv} S,$$

$$(9) q \in S + t.$$

(7) and (9) show that p and q can be connected in the interior of S + t by some Jordan arc  $\gamma$ . From (5) and (8) one obtains that  $\gamma$  has a point, say x, in common with bdr conv S. Because of the assumption (3) it is clear that

$$(10) x \in S.$$

On the other hand, (5) and (8) show that  $x \neq p$ ,  $x \neq q$  and therefore

(11) 
$$x \in \operatorname{int}(S+t) = \operatorname{int} S_j.$$

Because of (10) and the strong connectivity of S there are interior points of S in any neighborhood of x. This together with (11) shows that for some  $j \neq 1$ 

int 
$$S_1 \cap \text{int } S_j \neq \emptyset$$

in contradiction to (2).

II. Assume that bdr conv  $S \not = S$ . This means that there exists a point g with  $g \in \text{bdr conv } S$  and

$$(12) g \notin S.$$

By a well-known version of the theorem of Carathéodory on the convex hull of connected sets (see Bonnesen-Fenchel [1], p. 9) there is a closed line segment  $\sigma = [s_1, s_2]$  with  $s_1 \in S$ ,  $s_2 \in S$  and g in its (relative) interior. If L denotes a support line for conv S which contains g, it is obvious that

(13) 
$$\sigma \subset L$$
.

Let H be the halfplane which is bounded by L and contains conv S, and let  $H_i$  be defined by  $H_i = H + t_i$  where  $t_i$  is the translation vector determined by  $S_i = S + t_i$ . Then, the union of all the halfplanes  $H_i$  is again one of these halfplanes, say  $H_k$ . Since  $H_k$  contains every  $S_i$  it follows that the line  $L_k = L + t_k$  is a support line of K. By a proper assignment of the subscripts it can be achieved that k = 1 and therefore  $L_k = L$ . Hence, there is no loss in generality by assuming that the line L which contains  $\sigma$  is a support line of K. This implies in particular that

(14) 
$$\sigma \subset \operatorname{bdr} K$$
.

Because of the strong connectivity of S it is possible to connect the points  $s_1$  and  $s_2$  in the interior of S by some Jordan arc  $\tau$ . Since (13) implies that  $\sigma$  contains no interior points of S the arcs  $\sigma$  and  $\tau$  have only the points  $s_1$  and  $s_2$  in common. Let  $\lambda$  be the closed Jordan curve composed of  $\sigma$  and  $\tau$ . Then, the Jordan curve theorem shows that the complement of  $\lambda$  (with respect to  $R^2$ ) consists of two open connected sets which have the same boundary, namely  $\lambda$ . Further, one of these regions, say J, is bounded and the other is unbounded.

From the inclusions  $J \subset (J \cup \lambda) \subset \text{conv}(J \cup \lambda) = \text{conv}\lambda$  and  $\lambda = (\sigma \cup \tau) \subset ((\text{conv}\,S) \cup S) \subset \text{conv}\,S \subset K$  it follows immediately that

$$(15) J \subset \operatorname{conv} S$$

and

$$(16) J \subset K.$$

Because of (12) and the compactness of S it is obvious that the point g has positive distance from S. Using the fact that  $g \in \lambda = \text{bdr } J$  one can find a point q in J which is so close to g that  $q \notin S$ . This, together with (1) and (16) shows that q is contained in some  $S_h \neq S$ . Actually, one may assume that q is in the interior of  $S_h$ . If necessary this can be achieved by a sufficiently small change in the selection of q without disturbing the relations  $q \in J$ ,  $q \notin S$ . On the other hand, there is a point p with  $p \in \text{int } S_h$  and  $p \notin J$ . If such a point would not exist one had int  $S_h \subset J$ . But then (15) shows that int  $S_h \subset \text{conv } S$  and by taking the closure and the convex hull one would obtain conv  $S_h = (\text{conv } S) + t_h \subset \text{conv } S$  with  $t_h \neq 0$  and this is certainly impossible. Note that the closure of the interior of  $S_h$  is  $S_h$  since the strong connectivity implies that there are interior points in any neighborhood of a boundary point.

Hence, it has been found that there are points p, q with the following properties:

$$(17) p \in \operatorname{int} S_h, \ q \in \operatorname{int} S_h,$$

$$(18) p \notin J, \ q \in J.$$

Let  $\kappa$  be a Jordan arc which connects p and q in the interior of  $S_h$ . Because of (17) the endpoints of  $\kappa$  are also in int  $S_h$  and therefore in int K. This fact, if compared with (14), shows that  $\kappa$  and  $\sigma$  are disjoint. On the other hand, it follows from (18) that  $\kappa$  must contain a point of bdr  $J = \lambda = \tau \cup \sigma$ . Writing  $\tau' = \tau - \{ s_1, s_2 \}$  it has therefore been shown that  $\kappa$  and  $\tau'$  have a point, say  $\kappa$ , in common. But this implies that  $\kappa \in \kappa \subset K$  and  $\kappa \in K$  and  $\kappa \in$ 

## REFERENCES

- [1] Bonnesen, T. und W. Fenchel. *Theorie der konvexen Körper*. Ergebn. d. Math., Bd. 3. Berlin-Göttingen-Heidelberg, Springer 1934.
- [2] Groemer, H. Über translative Zerlegungen konvexer Körper. Arch. d. Math. 19 (1968), 445-448.

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