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A CONSTRUCTION OF GAUSS

by C. W. BARNES

1. Introduction

Every prime of the form 4n + 1 can be expressed uniquely as the sum of two squares. Suppose $p = x^2 + y^2$ where p is a prime of the form 4n + 1. A construction for x and y was given by Legendre [8] in terms of the continued fraction for \sqrt{p} . In [1] we gave a new construction for x and y, again using the continued fraction for \sqrt{p} . A summary of the various constructions is given in Davenport [5], pages 120-123.

Gauss [6] remarked that if p = 4n + 1, and if α and β are defined by

$$\beta \equiv \frac{(2n)!}{2(n!)^2} \pmod{p}, \ \alpha \equiv (2n)! \ \beta \pmod{p}, \text{ where } |\alpha| < \frac{p}{2}, |\beta| < \frac{p}{2} \text{ then}$$

 $p = \alpha^2 + \beta^2$; a particularly simple construction to state. Proofs of the construction of Gauss were given by Cauchy [4], page 414, and Jacobsthal [7]; however, neither of them is simple.

In the present note we give a simple proof of the construction of Gauss based on the method in [1].

2. CONTINUED FRACTIONS

We continue with the notation in [1]. The results we need can be found in Perron [9]. We denote the simple continued fraction

$$\begin{array}{c}
 a_0 + \underline{1} \\
 a_1 + \underline{1} \\
 a_2 + \underline{1}
\end{array}$$

$$+\frac{1}{a_n}$$

by $[a_0, a_1, ..., a_n]$. For $0 \le m \le n$ we denote the numerator and denominator of the m^{th} approximant to $[a_0, a_1, ..., a_n]$ by A_m and B_m respectively.

If p is a prime of the form 4n + 1, then

(2)
$$\sqrt{p} = [a_0, \overline{a_1, ..., a_m, a_m, ..., a_1, 2a_0}]$$

in the usual notation for a periodic continued fraction. The symmetric part of the period does not have a central term. In [1] we proved that $p = x^2 + y^2$ where

$$(3) x = pB_m B_{m-1} - A_m A_{m-1}$$

$$y = A_m^2 - pB_m^2$$

and where $\frac{A_m}{B_m}$ is the m^{th} approximant to (2). We also showed that

(5)
$$p = \frac{A_m^2 + A_{m-1}^2}{B_m^2 + B_{m-1}^2}.$$

3. THE QUADRATIC CHARACTER OF

$$\frac{(2n)!}{2(n!)^2}.$$

It is well known that if p is a prime of the form 4n + 1 then $\left\{ \left(\frac{p-1}{2} \right)! \right\}^2$ $\equiv -1 \pmod{p}$; that is, $(2n)!^2 \equiv -1 \pmod{p}$. We make use of this in the Lemma. If p = 4n + 1 is a prime then $\frac{(2n)!}{2(n!)^2}$ is a quadratic residue of p.

Proof. We use Euler's criterion. Thus if we suppose that $\frac{(2n)!}{2(n!)^2}$ is a quadratic nonresidue of p we have $\left\{\frac{(2n)!}{2(n!)^2}\right\}^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ and thus $\left\{(2n)!^2\right\}^{\frac{p-1}{4}} \equiv -\left\{2(n!)^2\right\}^{\frac{p-1}{2}} \pmod{p}$. Since $(2n)!^2 \equiv -1 \pmod{p}$ and $n!^{p-1} \equiv 1 \pmod{p}$ we have $(-1)^n \equiv -2^{\frac{p-1}{2}} \pmod{p}$, or $(-1)^{n+1} \equiv (-1)^{\frac{p^2+1}{8}}$, using the standard result for the quadratic character of 2 with res-

pect to an odd prime. We finally get $(-1)^{n+1} \equiv (-1)^{2n^2+n}$ or $(-1)^{n+1} \equiv (-1)^n \pmod{p}$ which is a contradiction since p is an odd prime. Thus $\frac{(2n)!}{2(n!)^2}$ is a quadratic residue of p.

4. The Construction of Gauss

THEOREM. Suppose p=4 n+1 is a prime and $p=x^2+y^2$ where x and y are given by (3) and (4). Let β and α denote respectively the numerically smallest residues of $\frac{(2n)!}{2(n!)^2}$ and (2n)! β modulo p, so that $|\alpha|<\frac{p}{2}$, $|\beta|<\frac{p}{2}$. Then $p=\alpha^2+\beta^2$.

Proof. By (5) we have, using the remark at the beginning of section 3, $A_m^2 + A_{m-1}^2 \equiv 0 \pmod{p}$ and hence $-A_m^2 \equiv A_{m-1}^2 \pmod{p}$, so that $\{(2n)!\}^2 A_m^2 \equiv A_{m-1}^2 \pmod{p}$, and since p is a prime $(2n)! A_m \equiv \pm A_{m-1} \pmod{p}$. Supposing the negative sign holds we have $(2n)! A_m^2 \equiv -A_m A_{m-1} \pmod{p}$. Therefore we obtain $(2n)! A_m^2 - (2n)! p B_m^2 \equiv (p B_m B_{m-1} - A_m A_{m-1}) \pmod{p}$, so that by (3) and (4) we get

(6)
$$x \equiv (2n) ! y \pmod{p}.$$

If the positive sign holds above it follows that $x \equiv -(2n) ! y \pmod{p}$ which is just as good for our present purposes since we are not concerned with the signs of x and y. We will comment on the signs in section 5.

By the lemma we have $\{\frac{(2n)!}{2(n!)^2}\}^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ so $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}}$ (n!) $^{p-1}$ (mod p), and therefore $(2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}}$ (mod p) since (n!, p) = 1. We have $x \equiv \pm (2n)! y \pmod{p}$, and since each of y and -1 is a quadratic residue of p, $x^{\frac{p-1}{2}} \equiv (2n)!^{\frac{p-1}{2}} \equiv 2^{\frac{p-1}{2}}$ (mod p), and in terms of the Legendre symbol it follows that $(\frac{x}{p}) = (\frac{2}{p})$; that is, the quadratic character of x with respect to x is the same as the quadratic character of x with respect to x is the same as the quadratic character of x with respect to x.

Suppose 2 is a quadratic residue of p. Then

 $2^{\frac{p-1}{2}}(n!)^{p-1}(A_mA_m-1)^{\frac{p-1}{2}} \equiv (A_mA_{m-1})^{\frac{p-1}{2}} \equiv (-x)^{\frac{p-1}{2}} \equiv x^{\frac{p-1}{2}} \equiv 1 \pmod{p}.$

Next, if 2 is a quadratic nonresidue of p we have

$$2^{\frac{p-1}{2}}(n!)^{p-1}(A_mA_{m-1})^{\frac{p-1}{2}} \equiv -(-x)^{\frac{p-1}{2}} \equiv -(x)^{\frac{p-1}{2}} \equiv -(-1) \equiv 1$$
 (mod p),

and we conclude that $2(n!)^2 A_m A_{m-1}$ is a quadratic residue of p. By (3), (4), and (6) we have

$$(2n) ! y \equiv -A_m A_{m-1} \pmod{p},$$

$$2(n !)^2 (2n) ! y \equiv -2(n !)^2 A_m A_{m-1} \pmod{p}$$

and

$$-2(n!)^2(2n)!y \equiv b^2 \pmod{p}$$

for some quadratic residue b^2 . Therefore

$$-2(n !)^{2} (2n) ! y \equiv -(2n) !^{2} b^{2} \pmod{p},$$

$$-2(n !)^{2} y \equiv -(2n) ! b^{2} \pmod{p},$$

and finally

$$y \equiv \frac{(2n)!}{2(n!)^2} b^2 \pmod{p}.$$

Hence by (6)

$$x \equiv \frac{(2n)!^2}{2(n!)} b^2 \pmod{p}.$$

Let $b^2 \equiv r \pmod{p}$, $|r| < \frac{p}{2}$, so that (r, p) = 1. Then in terms of α ,

 β , and r, $x \equiv \alpha r \pmod{p}$ and $y \equiv \beta r \pmod{p}$. There are unique integers K and L such that $x = \alpha r + Kp$, $y = \beta r + Lp$. Then

$$x^2 + y^2 = (\alpha^2 + \beta^2) r^2 + (K^2 + L^2) p^2 + 2rp(\alpha K + \beta L),$$

or

$$p = (\alpha^2 + \beta^2) r^2 + (K^2 + L^2) p^2 + 2rp(\alpha K + \beta L).$$

Suppose that $|r| > 1, K \neq 0$, and $L \neq 0$. The last equation can be written

(7)
$$pK^2 + (2r\alpha p)K + \{L^2p^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p\} = 0$$
.

Since (7) is a quadratic in K and we are supposing that the integral root is not zero we have

$$K \mid \{ L^2 p^2 + 2r\beta pL + (\alpha^2 + \beta^2) r^2 - p \}.$$

There is an integer t such that

$$L^{2}p^{2} + 2r\beta pL + (\alpha^{2} + \beta^{2})r^{2} - p = Kt$$

and therefore (7) vanishes when

$$K = \frac{L^2 p^2 + 2r\beta p L + (\alpha^2 + \beta^2) r^2 - p}{t}.$$

That is

(8)
$$\{ L^2 p^2 + 2r\beta pL + (\alpha^2 + \beta^2) r^2 - p \} \{ t^2 + 2r\alpha pt + p \{ L^2 p^2 + 2r\beta pL + (\alpha^2 + \beta^2) r^2 - p \} \} = 0$$

The discriminant of the quadratic function

$$t^2 + 2r\alpha pt + p \{ p^2L^2 + 2r\beta pL + (\alpha^2 + \beta^2)r^2 - p \}$$

is $4p^2 \{ p - (pL + \beta r)^2 \}$ which is not zero. It follows that the second factor in (8) cannot be zero; otherwise we would have two distinct integral values for t giving rise to two distinct integers K, whereas K is unique. Hence we have

(9)
$$p^{2}L^{2} + 2r\beta pL + (\alpha^{2} + \beta^{2})r^{2} - p = 0$$

and since we are supposing that $L \neq 0$, we see that

 $L \mid \{ (\alpha^2 + \beta^2) r^2 - p \}$ so that for an integer u we have $(\alpha^2 + \beta^2) r^2 - p$ = L u and (9) vanishes when

$$L = \frac{(\alpha^2 + \beta^2) r^2 - p}{u},$$

so that

$$(10) \left\{ (\alpha^2 + \beta^2) r^2 - p \right\} \left\{ u^2 + 2r\beta pu + p^2 \left\{ (\alpha^2 + \beta^2) r^2 - p \right\} \right\} = 0.$$

As before we consider the quadratic function

$$u^{2} + 2r\beta pu + p^{2} \{ (\alpha^{2} + \beta^{2}) r^{2} - p \}$$

The discriminant is $4p^2(p-\alpha^2r^2)$ which cannot vanish, so that, as before, the first factor in (10) must be zero, and we have

(11)
$$(\alpha^2 + \beta^2) r^2 - p = 0$$

which is a contradiction since $\alpha^2 + \beta^2 > 1$ and we are supposing that |r| > 1.

Therefore we cannot have |r| > 1, $K \ne 0$, and $L \ne 0$. If |r| = 1 we see that K = L = 0 since $|x - \alpha r| < p$ and $|y - \beta r| < p$ in this case. If |r| > 1 with K = L = 0 we would have $x = \alpha r$, $y = \beta r$ and hence (x, y) > 1, whereas x and y are relatively prime. Finally it remains to consider the possibility of having |r| > 1 with one of K and L zero, the other nonzero. This if we suppose that |r| > 1, K = 0, $L \ne 0$, we obtain (9) which, as we have seen, leads to a contradiction. On the other hand the supposition that |r| > 1 with $K \ne 0$, L = 0 implies that (11) would hold with $r^2 > 1$.

We conclude that |r| = 1, K = 0 and L = 0. Hence $x = \pm \alpha$, $y = \pm \beta$ and $\alpha^2 + \beta^2 = p$.

In [1], Corollary 2, we observed that if $p = x^2 + y^2$ then, in our notation, y is a quadratic residue of p. Collecting our results we have the

COROLLARY. Let $p = x^2 + y^2$ where p is a prime of the form 4n + 1 with x and y given by (3) and (4). Then $\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right)$ and $\left(\frac{y}{p}\right) = 1$.

5. Conclusion

We saw that $x = \pm \alpha$, $y = \pm \beta$. When p = 13 we have y = -3, $\beta = -3$; when p = 29, y = -5, $\beta = 5$, and when p = 41, y = 5, $\beta = 5$. Hence the sign of y, determined by the approximants to a continued fraction depends on the integer m, the number of terms in the finite segment of (2) which is used, can agree with that of β or be opposite that of β . The same applies to x and α . In [1], Theorem 1, we gave a construction which always gives positive values for x and y. Other various constructions, as we have seen, do not have this property.

Finally we comment on the numbers $\frac{(2n)!}{2(n!)^2}$ which we denote by a_n for n = 1, 2, 3, ...

The members of the sequence $\{a_n\}$ are related to the numbers $b_{n+1} = \frac{(2n)!}{(n+1)!n!}$, n = 0, 1, 2, ..., which, as mentioned by Becker [2], have a variety of applications. Birkhoff [3] pointed out that b_n is an integer for every positive integer n, and noted the recurrence relation $b_n = \sum_{i=1}^{n-1} b_i b_{n-i}$;

a relation which was also obtained by Wedderburn [10]. The results of this note depend on the fact that a_n is an integer, at least when p = 4n + 1 is a prime. Although it is known that a_n is an integer for every positive integer n, we can see that this also follows readily from [3]. For we have $2 a_n = (n+1) b_{n+1}$. If n is even, it follows that b_{n+1} is even since (2, n+1) = 1. Therefore $a_n = (n+1) \frac{b_{n+1}}{2}$ is an integer. If n is odd

then $2 \mid (n+1)$ and in this case also $a_n = \frac{n+1}{2} b_{n+1}$ is an integer. A list of values for a_n can be obtained from the second column of a table in [2], page 699, headed N_n , by multiplying the (n+1)st member by $\frac{n+1}{2}$.

REFERENCES

- [1] Barnes, C. W. The Representation of Primes of the Form 4n + 1 as the Sum of Two Squares, L'Enseignement Mathématique, 18 (1972), pp. 289-299.
- [2] Becker, H. W. Discussion of Problem 4277, American Mathematical Monthly, 56 (1949), pp. 697-699.
- [3] Birkhoff, Garrett. Problem 3674, American Mathematical Monthly, 42 (1935), pp. 518-521.
- [4] CAUCHY, A. Sur les Formes quadratiques de certaines Puissances des Nombres Premiers ou du quadruple de ces Puissances, Œuvres complètes 1re série, tome 3, pp. 390-437.
- [5] DAVENPORT, H. The Higher Arithmetic, Hutchinson's University Library, London, 1952.
- [6] Gauss, C. F. Werke Bd. 2, S. 90-91.
- [7] JACOBSTHAL, Ernst. Über die Darstellung der Primzahlen der Form 4n + 1 als Summe Zweier Quadrate, Journal für die Reine und Angewandte Mathematik, Band 132, (1907), pp. 238-245.
- [8] LEGENDRE, A. M. Théorie des Nombres. Troisième édition, Paris, 1830.
- [9] Perron, Oskar. Die Lehre von den Kettenbrüchen. Chelsea, New York, 1951.
- [10] WEDDERBURN, J. H. M. The Functional Equation $g(x^2) = 2 \alpha x + [g(x)]^2$, Annals of Mathematics, (2), 24, (1922-1923), pp. 121-140.

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