

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 20 (1974)
Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: AN IMMERSION OF $T^n - D^n$ INTO R^n
Autor: Ferry, Steven
DOI: <https://doi.org/10.5169/seals-46901>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 08.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

AN IMMERSION OF $T^n - D^n$ INTO R^n

by Steven FERRY

Let $T^n = S^1 \times \dots \times S^1$ be the n -torus and let $D^n \subset T^n$ be an embedded disc. Kirby, Siebenmann, and Edwards use immersions of $T^n - D^n$ into R^n repeatedly in their work on stable homeomorphisms and triangulations. Although the existence of such immersions follows trivially from the work of Smale and Hirsch, it is appealing to have a more elementary construction. We will provide an explicit formula. Less explicit constructions have been given by D. Barden and J. Milnor.

Elements of T^n will be written as $(\theta_1, \dots, \theta_n)$ with $\theta_1 \in S^1$. θ_1 will also be thought of as a real number $0 \leq \theta < 2\pi$. Occasionally $(\theta_1, \dots, \theta_n)$ will be denoted by θ .

LEMMA 1. Let $(\theta, t) \rightarrow (f_1(\theta, t), \dots, f_{n+1}(\theta, t))$ be an embedding of $T^n \times (-1, 1)$ into R^{n+1} such that $f_{n+1}(\theta, t) > 0$. The map of $T^{n+1} \times (-1, 1)$ into R^{n+2} defined by

$$(\theta, t) \rightarrow (f_1(\theta, t), \dots, f_n(\theta, t), f_{n+1}(\theta, t) \cos \theta_{n+1}, f_{n+1}(\theta, t) \sin \theta_{n+1})$$

is an embedding.

We define the *standard embedding* of $T^n \times (-1, 1)$ into R^{n+1} to be the embedding obtained by starting with $(\theta_1, t) \rightarrow ((1+t) \cos \theta_1, (1+t) \sin \theta_1 + 2)$ and iterating the process described in lemma 1. At each stage we must add 2^n to the last term so that the condition $f_{n+1}(\theta, t) > 0$ will be satisfied. For example, in the standard embedding of $T^3 \times (-1, 1) \rightarrow R^4$ we have

$$f_3(\theta, t) = (((1+t) \sin \theta_1 + 2) \sin \theta_2 + 4) \cos \theta_3$$

and

$$f_4(\theta, t) = (((1+t) \sin \theta_1 + 2) \sin \theta_2 + 4) \sin \theta_3 + 8.$$

Let $S = \{ \theta \in T^n \mid \theta_1 = 0 \text{ for some } i, 1 \leq i \leq n \}$.

Let $\varphi: T^n \rightarrow R^1$ be defined by

$$\varphi(\theta) = \frac{\sin \theta_1 \dots \sin \theta_n}{2^n} + \frac{\sin \theta_2 \dots \sin \theta_n}{2^{n-1}} + \dots + \frac{\sin \theta_n}{2}$$

Theorem 1. Let $(\theta, t) \rightarrow (f_1(\theta, t), \dots, f_{n+1}(\theta, t))$ be the standard embedding of $T^n \times (-1, 1)$ into R^{n+1} . For some $\varepsilon > 0$ the map $\theta \rightarrow$

$\rightarrow (f_1(\theta, \varepsilon\varphi(\theta), \dots, f_n(\theta, \varepsilon\varphi(\theta)))$ has nonsingular Jacobian on S . It therefore immerses a regular neighborhood of S (i.e. $T^n - D^n$) into R^n .

Proof. Using elementary properties of determinants, we compute:

$$\begin{aligned} \det \left(\frac{\partial f_i}{\partial \theta_j} + \varepsilon \frac{\partial f_i}{\partial t} \frac{\partial \varphi}{\partial \theta_j} \right) \Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} &= \det \left(\begin{array}{c|c} \frac{\partial f_i}{\partial \theta_j} + \varepsilon \frac{\partial f_i}{\partial t} \frac{\partial \varphi}{\partial \theta_j} & 0 \\ \hline \frac{\partial \varphi}{\partial \theta_j} & 1 \end{array} \right) \Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = \\ &= \det \left(\begin{array}{c|c} \frac{\partial f_i}{\partial \theta_j} & -\varepsilon \frac{\partial f_i}{\partial t} \\ \hline \frac{\partial \varphi}{\partial \theta_j} & 1 \end{array} \right) \Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = \det \left(\begin{array}{c|c} \frac{\partial f_i}{\partial \theta_j} & -\varepsilon \frac{\partial f_i}{\partial t} \\ \hline \frac{\partial \varphi}{\partial \theta_j} & 0 \end{array} \right) \Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} + \det \left(\begin{array}{c|c} \frac{\partial f_i}{\partial \theta_j} & -\varepsilon \frac{\partial f_i}{\partial t} \\ \hline 0 & 1 \end{array} \right) \Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} \end{aligned}$$

By construction, f_i involves only $\theta_1, \dots, \theta_i$ and $\frac{\partial f_i}{\partial \theta_i}$ has a factor of $\sin \theta_i$.

Thus, on S the upper left hand corner of the second matrix is triangular with at least one zero on the diagonal. We have

$$\det \left(\frac{\partial f_i}{\partial \theta_j} + \varepsilon \frac{\partial f_i}{\partial t} \frac{\partial \varphi}{\partial \theta_j} \right) \Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}} = -\varepsilon \det \left(\begin{array}{c|c} \frac{\partial f_i}{\partial \theta_j} & \frac{\partial f_i}{\partial t} \\ \hline \frac{\partial \varphi}{\partial \theta_j} & 0 \end{array} \right) \Bigg|_{\substack{\theta \in S \\ t = \varepsilon\varphi}}$$

Notice that $f_{n+1}(\theta, 0) = 2^n \varphi(\theta)$ and that $\frac{\partial f_{n+1}}{\partial t}$ is identically zero

on S . Thus, if the above determinant is evaluated at $\theta \in S, t = 0$ it is $\left(\frac{-\varepsilon}{2^n}\right)$ times the determinant of the Jacobian of the standard embedding. It is therefore nonsingular when evaluated at $\theta \in S, t = \varepsilon\varphi$ for sufficiently small ε . This completes the proof.

In essence, we have perturbed the image of $T^n \times 0$ in R^{n+1} along its normal bundle so that projection into R^n is an immersion on S . More precise calculations show that ε may be taken to be 1.

Steven Ferry

Department of Mathematics
University of Kentucky
Lexington, KY. 40506

(Reçu le 25 septembre 1973)